Cpt 4. Actions of f.d. Hopf algs. and smash product.

0. abstract.

①. $G \overset{\text{act}}{\curvearrowright} S \xrightarrow{\text{analogy}} H \curvearrowright A$

②. classical question: 1. Integrality of A over A^H

2. finite generation of A^H

3. A^H-module structure of A

Note: Cpt 8: when $A^H \subseteq A$ is a Galois extension.

③. skew group ring $A * G \xrightarrow{\text{generalize}} A \# H$

relationship between A^H and $A \# H$

1. some background.

① Group ring

Let R be a (unital) ring and G a group. Denote

$RG = \{ f: G \to R \mid f$ is of finite support, i.e. $\#\{x : f(x) \neq 0\} < \infty \}$

Then the group ring RG is a free R-module and a ring via

(1) addition: $(f+g)(x) = f(x) + g(x)$

(2). module: $(\alpha \cdot f)(x) = \alpha \cdot (f(x))$

(3) multiplication: $f * g(x) = \sum_{uv=x} f(u) \cdot g(v)$,

Note: 1. $f * g$ is well-defined since f is of finite support }

2. $\{ x_a \mid a \in G, \forall b \in G, x_a(b) = \delta_{ab} \}$ is a basis of RG, and

$x_a * x_b = x_{a \cdot b}$ since $x_a * x_b(x) = \sum_{u \cdot v = x} \delta_{au} \cdot \delta_{bv} = \delta_{ab, x}$

3. when R is commutative, RG is a group algebra.

② Skew group ring. (analogous to semidirect product of groups).

Let R be a ring and G a finite group. Let $\varphi: G \to \mathrm{Aut}(R)$ be a group

homomorphism, the skew group ring of G over R induced by φ is the ring of formal sums $R \rtimes_\varphi G = \{ \sum_{g\in G} a_g \cdot g : a_g \in R \}$ via

(1) addition: $\sum_{g\in G} a_g \cdot g + \sum_{g\in G} b_g \cdot g = \sum_{g\in G} (a_g + b_g)\cdot g$

(2) product: $ag \cdot bh = a\varphi(g)(b)\cdot gh$

Note: let $\varphi(G) = I_R$, then $R\rtimes_\varphi G \cong RG$ is a group ring.

③. free product

$G * H = \{ g_1 h_1 \cdots g_r h_r \mid r \in \mathbb{Z}_+,\ g_i \in G,\ h_i \in H \}$

④. integral

Let B be a commutative ring, and A a subring of B. An element $b\in B$ is integral over A if $\exists$ monic polynoimal $f \in A[x]$, s.t. $f(b)=0$. If every element of B is integral over A, then we say that B is intergral over A, or equivalently, B is an integral extension of A.

Cpt 4.1 Mod/comod algs. smash product.

1. definitions

①. An algebra (A, m, u) is a left H-module algebra if

(1). A is a left H-module via $\gamma: H\otimes A \to A$, $h\otimes a \mapsto h\cdot a$

(2) m and u are H-module morphisms.

i.e. $h\cdot(ab) = (h_1\cdot a)(h_2 \cdot a)$, $h\cdot 1_A = \epsilon(h)\cdot 1_A$, $\forall\, h\in H,\ a, b\in A$.

diagram:

$$\begin{tikzcd}
H\otimes H\otimes A\otimes A \arrow[d, "I_H\otimes\tau\otimes I_A"] & H\otimes A\otimes A \text{ (start)} \arrow[l, "\Delta_H\otimes I_{A\otimes A}"'] \arrow[r, "I_H\otimes m"] \arrow[d, dashed] & H\otimes A \arrow[d, "\gamma"] \\
(H\otimes A)\otimes(H\otimes A) \arrow[r, "\gamma\otimes\gamma"] & A\otimes A \arrow[r, "m"] & A \text{ (end)}
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
& H\otimes \mathbb{K} \arrow[dl, "\gamma_{\mathbb{K}}"'] \arrow[dr, "I_H\otimes u"] & \\
\mathbb{K} \arrow[dr, "u"'] & & H\otimes A \arrow[dl, "\gamma"] \\
& A &
\end{tikzcd}$$

② Dually, an algebra (A, m, u) is a right H-comodule algebra if

(1) A is a right H-comodule via $\rho: A \to A\otimes H$

(2) m and u are H-comodule morphisms.

i.e. $(ab)_0 \otimes (ab)_1 = a_0 b_0 \otimes a_1 b_1$, $\rho(1) = 1\otimes 1$, $\forall\, a, b\in A$.

diagram:
$$\begin{tikzcd}
(A\otimes A)\otimes H\otimes H \arrow[r, "1_{A\otimes A}\otimes m_H"] & A\otimes A\otimes H \arrow[r, "m"] & A\otimes H \ (\text{end}) \\
(A\otimes H)\otimes(A\otimes H) \arrow[u, "1_A\otimes\tau\otimes 1_H"] & A\otimes A \ (\text{start}) \arrow[l, "\rho\otimes\rho"'] \arrow[u, dashed] \arrow[r, "m"'] & A \arrow[u, "\rho"']
\end{tikzcd}$$

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Remark: When H is f.d. A is a left H-module $\Leftrightarrow$ A is a right H^*-comodule.

Note: 1. M is a right C-comodule $\Rightarrow$ M is a left C^*-module via $c^*m = m_0\cdot c^*(m_1)$

M is a left A-module $\Rightarrow$ M is a right A^*-comodule via $m_1^*(a)\cdot m_0 = a\cdot m$

2. $f: M\to M\otimes V \Rightarrow \tilde{f}: V^*\otimes M\to M$, $v^*: M\otimes V\to M$

$v^*\otimes m \mapsto v^*(f(m))$ $\qquad$ $m\otimes v\mapsto v^*(v)\cdot m$.

"partial dual" also retains the commutativity of diagrams.

③. Let A be a left H-module algebra. Then the smash product algebra $A\# H$ is defined as follows, for all $a,b\in A$, $h,k\in H$:

(a) $A\# H\cong A\otimes H$ as $\mathbb{k}$-vector space.

(b). $(a\# h)(b\# k) = a(h_1\cdot b)\# h_2 k$

Note: 1. $A\hookrightarrow A\# 1\subseteq A\# H$ and $H\hookrightarrow 1\# H\subseteq A\# H$, since $a\# 1\cdot b\# 1 = ab\# 1$, and

$1\# h\cdot 1\# g = 1\cdot(h_1\cdot 1)\# h_2\cdot g = 1\#\varepsilon(h_1)h_2 g = 1\# hg$. Thus A and H are subalgs of $A\# H$.

2. Since $a\# 1\cdot 1\# h = a\# h$, we abbreviate $a\# h$ by ah. In this notation, $ha = (h_1\cdot a)h_2$

2. examples

①. Let A be a trivial H-modules, then $A\# H = A\otimes H$ as algebras.

pf: $(ah)(bk) = a(h_1\cdot b)h_2 k = ab\,\varepsilon(h_1)h_2 k = ab\,hk$

②. 1. Let $H=\mathbb{k}G$, and let A be an H-module algebra, then $g\cdot(ab) = (g\cdot a)(g\cdot b)$, $\forall g\in G$, and thus g acts as an endomorphism of A. In fact, $\forall g\in G(H)$, g acts as an automorphism of A. Thus we have a group morphism $G\to \mathrm{Aut}_{\mathbb{k}}A$

2. Conversely, any such map make A into a $\mathbb{k}G$-module algebra.

Note: $G \overset{\text{linearly}}{\curvearrowright} A \Leftrightarrow \mathbb{k}G \overset{\text{mod alg}}{\curvearrowright} A$.

In this case, $A\#\mathbb{k}G = A * G$, the skew group ring. $(ag)(bh) = a(g\cdot b)gh$

Note: 1. Hence # is a generalization of semi-product. | $\mathbb{K}G \otimes \mathbb{K}L = \mathbb{K}(G\times L)$

2. Semi-product : $(x_1, y_1)\cdot(x_2, y_2) = (x_1 x_2^{y_1}, y_1 y_2)$ | $\mathbb{K}L \# \mathbb{K}G = \mathbb{K}(L \rtimes G)$?

③. 1. Let $H = \mathbb{K}G$, but now let A be a $\mathbb{K}G$-comodule algebra. Then $A = \bigoplus_{g\in G} A_g$ is a G-graded vector space, where $A_g = \{a \in A : \rho(a) = a\otimes g\}$. Thus $\rho(a_g b_h) = (a_g \otimes g)(b_h \otimes h) = a_g b_h \otimes gh$, i.e. $a_g b_h \in A_{gh}$. Hence A is a G-graded algebra.

Note: $H = \mathbb{K}G \Rightarrow$ mod alg = skew group ring, comod alg. = graded alg.

2. When $|G| < \infty$. A is G-graded $\Leftrightarrow$ $\mathbb{K}G$-comod. alg $\Leftrightarrow$ A is a $(\mathbb{K}G)^*$-mod alg.

More precisely, $g^*\cdot a_h = a_h \cdot g^*(h) = \delta_{gh} a_h$; i.e. $\{g^*\}$ act as projections.

In this case, the multiplication in $A\#(\mathbb{K}G)^*$ is given by

$$(a\# g^*)(b\# h^*) = \sum_{uv=g} a(u^*\cdot b)\# v^* h^* = a\, b_{gh^{-1}} \# h^* \quad (v=h,\ u=gh^{-1})$$

where $b_{gh^{-1}}$ is the projection of b to $A_{gh^{-1}}$

④. 1. $\forall H$, consider $x\in P(H)$, the primitive elements, and let A be an H-module alg. Since $\Delta(x) = x\otimes 1 + 1\otimes x$, $x\cdot(ab) = (x\cdot a)b + a(x\cdot b)$; i.e. x acts as a $\mathbb{K}$-derivation of A and we have a Lie morphism $P(H) \to \mathrm{Der}_{\mathbb{K}}(A)$.

Note: $G(C)$ is a set while $G(B)$ is a group. Similarly, $P(B)$ is a Lie algebra.

Pf: 1. $g, h \in G(B) \Rightarrow g\cdot h \in G(B)$

2. $g, h \in P(B)$. Let $[gh] = gh - hg$, then $\Delta([gh]) = [gh]\otimes 1 + 1\otimes [gh]$, hence $[gh] \in P(B)$.

2. More generally, $x \in P_{g,h}(H) \Rightarrow x\cdot(ab) = (x\cdot a)(g\cdot b) + (h\cdot a)(x\cdot b)$, where g, h act as automorphism of A. We say that x acts as a g,h-derivation of A; it is also called a skew derivation.

Note: $x\in G(H) \Rightarrow x$ acts as automorphism of A (algebraic)

$x \in P(H) \Rightarrow x$ acts as derivation.

3. When $H = U(\mathfrak{g})$, the action of $\mathfrak{g}$ determines $U(\mathfrak{g})$. In this case $A\# H$ is sometimes called the differential polynomial ring. In particular when $\mathfrak{g} = \mathbb{K}x$, x acts as derivation δ of A, then $A\# U(\mathfrak{g}) = A[x;\delta]$, the usual

Ore extension in which $xa = (x_1 \cdot a)x_2 = \delta(a) + ax$.

4\. A special case of this construction is the first Weyl algebra A_1.

let $A = \Bbbk[y]$, and $\delta = \frac{d}{dy}$, then $A_1 = \Bbbk[y][x;\delta] = \Bbbk\langle x, y : xy - yx = 1\rangle$

Note: 1. R is a ring (not necessarily commutative), $\sigma: R \to R$ is a ring morphism, and $\delta: R \to R$ is a σ-derivation of R, i.e. $\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1) r_2$.

Then the Ore extension $R[x;\sigma,\delta]$, also called a skew polynomial ring obtain by given $R[x]$ the multiplication: $xr = \sigma(r)x + \delta(r)$.

2\. If $\delta = 0$, then the Ore extension is denoted $R[x;\sigma]$. (σ 什么作用)

3\. If $\sigma = I_R$, then the Ore extension is denoted $R[x;\delta]$ and is called a differential polynomial ring.

5\. 1. H is itself an H-module algebra for both the left and right adjoint actions. ad_l and ad_r are examples of inner actions.

2\. For either adjoint action, $H \# H \cong H \otimes H$, though the action may be non-trivial (Cpt 7.3.3).

pf: $h \rightharpoonup k = h_1 k S h_2 \Rightarrow h \rightharpoonup xy = h_1 xy S h_2 = h_1 x S h_2 h_3 y S h_4 = (h_1 \rightharpoonup x) \cdot (h_2 \rightharpoonup y)$

$h \rightharpoonup 1 = h_1 S h_2 = \epsilon(h) \cdot 1$

Note: 1. $ad_l: G \to \mathrm{Inn}\, G$. grp homomorphism. $ad_l: L \to \mathrm{Der}\, L$ Lie homomorphism.

2\. H is not an H-module algebra via left action.

3\. Similarly, H is an H-comodule algebra via coadjoint action.

6\. H^* is a right H^*-comodule $\Rightarrow$ H^* is a left H-module. H-mod alg.

Cpt 4. $H \# H^* \cong \mathrm{End}_{\Bbbk} H$, ${}_{H\#H^*}\mathcal{M} \rightleftarrows {}_H\mathcal{M}^H$

$H \# H^*$ is called the Heisenberg double. $\mathcal{H}(H)$.

Note: A-bimodule $\Leftrightarrow$ $A \otimes A^{op}$ left module.

7\. $H = U_q(sl_2)$, $A = \mathbb{C}[x]$. H-mod alg in a unique way.

Cpt 4.2

Abstract: Generalize the well-known fact: $|G|<\infty$, A comm, $G \curvearrowright A \Rightarrow A \overset{\text{integral}}{\curvearrowright} A^G$

1. Integral over A^H

① Main thm:

Let H be f.d. cocomm. and let A be a comm. H-module algebra, Then A is integral over A^H

②. Let $M \in {}_{A\#H}M$ s.t. $M \cong A^{(n)}$ as left A-modules, and let $f: M \to M$ be an $A\#H$-module map. Then χ_f, the characteristic polynomial of f in $\mathrm{End}_A(M)$, has coefficients in A^H

pf: 1. We first consider the case when $n=1$. Thus we may write $M = Am_0$, $\exists m_0 \in M$.

Then $f(m_0) = am_0$, for some $a \in A$. $\forall m = bm_0 \in A$, $f(m) = f(bm_0) = bf(m_0)$

$= bam_0 = abm_0 = am$, i.e. $f(m) = am$, $\forall m \in M$. Thus $f = a \cdot I_M$, $a = \det_A f$

2. We claim that $a \in A^H$, i.e. $(h \cdot a) = \varepsilon(h) a$

Now $\forall h \in H, m \in M$, $a(hm) = f(hm) = hf(m) = ham = (h_1 \cdot a) h_2 m$

But then $(h \cdot a) m \overset{h = h_1 \varepsilon(h_2)}{=} (h_1 \cdot a) \varepsilon(h_2) m$

$= (h_1 \cdot a) h_2 S h_3 m$ ② $(h_1 \cdot a) h_2 m = ahm$

$= a h_1 S h_2 m$

$= \varepsilon(h) am$

Using $m = m_0$, and the freeness of M, we see $h \cdot a = \varepsilon(h) a$ for all $h \in H$. Thus $a \in A^H$, proving the case $n=1$.

? 3. Since M is a left $A\#H$ module and A is commutative, $T^n M$ is also a left $A\#H$ module ($h \cdot m \otimes n = h_1 m \otimes h_2 n$)

Let I be the A-submodule of $T^n M$ generated by symmetric tensors $\{m \otimes m \mid m \in M\}$. Since H is cocommutative, it stabilizes I; and thus I is an $A\#H$ module. Hence $N = \Lambda^n M = T^n M / I$

is a quotient $A\#H$-module of $T^m M$.

$f: M \to M$ induces the $A\#H$-module map $\Lambda^n f: N \to N$, with the same determinant as f, and N is free of rank 1 as an A-mod.

We may thus apply the $n=1$ case to see that $\det \Lambda^n f = \det f \in A^H$.

4. Let t be an indeterminante. Then $A[t]$ and $M[t]$ become $A\#H$-mods by letting H act trivially on t, and f extends to an $A\#H$ map $\tilde{f}: M[t] \to M[t]$. But now $t\cdot id - \tilde{f}$ is an $A\#H$-map, and thus $\det_A(t\cdot id - \tilde{f}) = \chi_f(t) \overset{??}{\in} A[t]^H = A^H[t]$ by the above arguments.

Note 1: 1. A is a left $A\#H$-module via $(b\#h) \to (a\#1) = b(h\cdot a)\#1$ $(\neq (b\#h)(a\#1))$

2. H dosen't need to be f.d.

Note 2: 1. Let M, N be left $A\#H$-modules and A be commutative, then $M \otimes_A N$ is an left $A\#H$-module

2. $\det \Lambda^n f = \det f$

3. A-modules lifts to $A\#H$-modules

Proof of the main thm:

Pf: Let $M = A\#H$; M is a free left A-module of rank $n = \dim H$

Let $f = r_a$, right multiplication by $a \in A$; f is a left $A\#H$-map (since left and right action are commutative). Thus by 4.2.2, $\chi_{r_a} \in A^H[t]$ and a is integral over A^H.

2. $\mathbb{K}$-affine.

① A $\mathbb{K}$-algebra A is called $\mathbb{K}$-affine if it's finitely generated as $\mathbb{K}$-algebra. i.e. $\exists \{a_i\}_{i=1}^n \subseteq A$ s.t. $A = \{ f(a_1, \cdots a_n) \mid f \in \mathbb{K}[x_1, \cdots, x_n] \}$.

②. Artin-Tate lemma.

Let $B \subseteq A$ be commutative $\mathbb{K}$-algebras. If A is $\mathbb{K}$ affine and integral over B, then B is $\mathbb{K}$-affine.

③ Thm: let H be a f.d. cocomm. hopf algebra, and A a comm. algebra such that $H \overset{\text{mod}}{\curvearrowright} A$ and A is $\mathbb{K}$-affine. Then A^H is $\mathbb{K}$-affine

pf: $A \overset{\text{integral}}{\curvearrowright} A^H$, $\xrightarrow{\text{Artin lemma}}$ A^H is $\mathbb{K}$-affine.

Q: A comm. H f.d $\overset{\text{mod}}{\curvearrowright} H \overset{?}{\Rightarrow} A \overset{\text{integral.}}{\curvearrowright} A^H$

Summary:

1. $K \rtimes H \Rightarrow k_1h_1 \cdot k_2h_2 = k_1 h_1 k_2 h_1^{-1} \cdot h_1h_2 = k_1 k_2^{h_1} \cdot h_1 h_2$, $H \overset{\text{grp. action}}{\curvearrowright} K$

$A \# H \Rightarrow ah\, bg = a(h_1 \cdot b) \# h_2 g$, $H \overset{\text{mod}}{\curvearrowright} A$

i.e. $\#$ is a generalization of semiproduct.

2. $A \# H \cong A^{(\dim H)}$ as free left A mod.

$\cong H^{(\dim A)}$ as free right H-mod.

Since $(a\#1)(1\#h) = ah$, writh $ah = a\#h$ for short.

3. Main thm.

H f.d. cocomm. A comm. $\Rightarrow A \overset{\text{integral}}{\curvearrowright} A^H$

$+$

A is $\mathbb{K}$-affine $\Rightarrow A^H$ is $\mathbb{K}$-affine

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Cpt 4.3. Trace functions and inv. (non-comm.)

0. Integrality of A over A^H.

① schelter-integral.

②. Que: $H \overset{\text{mod}}{\curvearrowright} A$, H f.d. s.s. $\overset{?}{\Rightarrow} A \overset{\text{schelter integral}}{\curvearrowright} A^H$

1. Main results (4.3.7)

let A be a left Noetherian which is an affine $\mathbb{K}$-algebra,

let H be f.d. and $H \curvearrowright A$ such that $\hat{t} : A \to A^H$ is surjective.

Then A^H is $\mathbb{K}$-affine (and Noetherian by 4.3.5)

2. lemmas

① trace function.

$tr: A \to A^G \Rightarrow \hat{t}: A \to A^H, \ t \in \int_H^l$

$a \mapsto \sum_{g \in G} g \cdot a \qquad a \mapsto t \cdot a$

$\hat{t}$ is an A^H-bimodule map.

Pf: $\forall b \in A^H$, $t \cdot ba = (t_1 \cdot b)(t_2 \cdot a) = \varepsilon(t_1) b (t_2 \cdot a) = b(t \cdot a)$

Hence $\hat{t}$ is a left A^H module. The argument of the right side is the same.

② Lemma: Assum that H is f.d. $H \curvearrowright A$ and $\hat{t}$ is surjective.

Then $\exists$ a non-zero idempotent $e \in A \# H$ s.t. $e(A\#H)e = A^H e \cong A^H$ as algebras.

Note: 1. $\hat{t}$ is surjective $\Leftrightarrow \exists c \in A$ s.t. $\hat{t}(c) = 1$, i.e. $t \cdot c = 1$.

2. $hat = (h \cdot a)t$

pf: $(1\#h)(a\#t) = (h_1 \cdot a) \# h_2 t = \varepsilon(h_2) h_1 \cdot a \# t = (h \cdot a) \# t$

Pf: Since $\hat{t}$ is surjective, there exists $c \in A$ with $\hat{t}(c) = t \cdot c = 1$.

Define $e = tc$, then $e^2 = \underline{tct}c \overset{\text{asso.}}{=} (t \cdot c)tc = tc = e$.

For any $a \in A$, $h \in H$, we then have

$e(ah)e = tca\underline{h}tc = \varepsilon(h)\, \underline{tcat}c \overset{\text{mod}}{=} \varepsilon(h) \cdot (t \cdot ca)tc \in A^H e$

Conversely, if $a \in A^H$, then $t \cdot (ca) = (t \cdot c)a = a$, and thus

$ae = atc = t \cdot (ca)tc = tcatc$. That is, $e(A\#H)e = A^H e$

Finally, this is algebra-isomorphic to A^H, since

$(ae)(be) = atcbtc = abe$ as above

Note: $\hat{t}$ is surjective $\Rightarrow e = tc$ is as required.

③ Corollary.

Assume that H f.d. $\curvearrowright A$ and $\hat{t}$ is surjective. If A is left or right Noetherian, then so is A^H.

Pf: (sketch).

1. A is left Noetherian, then so is $A\#H$

2. right part: 4.4.3 / 7.2.11.

3. If S is Noetherian, then eSe is Noetherian

3. Blowing up a chain of ideals of A^H to A, applying $\hat{t}$ to recover the original chain.

Remark: the Corollary is false if $\hat{t}$ is not surjective.

⑥. Let S be a $\mathbb{k}$-algebra and e a non-zero idempotent in S. If S is $\mathbb{k}$-affine and left Noetherian, then eSe is $\mathbb{k}$-affine.

pf:? (sketch).

1. Since S is left Noetherian, SeS is a f.g. left ideal of S.

2. We claim that eS is a f.g. left eSe-mod.

Let $SeS = \sum_{i=1}^{n} Sx_i$ where $x_i = \sum_j v_{ij} e w_{ij}$. For any $r \in S$, $er \in e(SeS)$ and so $er = e(\sum_i s_i x_i) = \sum_{i,j} e s_i v_{ij} e w_{ij}$. Thus the set $\{ew_{ij}\}$ generates eS as an eSe-module, proving the claim.

For simplicity, rewrite the generators as $\{ew_i\}$.

3. pass.

③ proof of the main thm.

A left Noetherian, $\mathbb{k}$-affine; H f.d. $\overset{\text{mod}}{\curvearrowright} A$, $\hat{t}$ surjective $\Rightarrow A^H$ is $\mathbb{k}$-affine

pf: 4.3.4 and 4.3.6. using that $S = A\#H$ is left Noetherian.

Remark: 1. 4.2.5: H f.d. cocomm. A comm. $\Rightarrow A^H$ $\mathbb{k}$-affine

4.3.7: H f.d. $\hat{t}$ surjective. A left Noetherian.

4.3.7 fails if $\hat{t}$ is not surjective or A is not left Noetherian.

2. H is s.s. $\Leftrightarrow \varepsilon(\int_H^l) \neq 0$

$\Rightarrow$ let $t \in \int_H^l$ s.t. $\varepsilon(t) = 1$, then $\hat{t}(1) = t \cdot 1 = 1$

$\Rightarrow \hat{t}$ is surjective.

4. surjective trace

① total integral.

Let A be an right H-comod alg. Then a right total integral for A is a right H-comod map $\varphi: H \to A$. s.t. $\varphi(1)=1$

② an observation of Radford.

$0 \neq T \in \int_{H^*}^{l}$, H^* is an right H^*-comod $\Rightarrow H^*$ is a left H-mod.

$$\theta: H \to H^* \quad \text{is a left } H\text{-mod isomorphism.}$$
$$h \mapsto (h \rightharpoonup T)$$

Setting $t = \theta^{-1}(\varepsilon)$, then $t \rightharpoonup T = \varepsilon$. Since $\theta^{-1}(t \rightharpoonup T) = t = \theta^{-1}(\varepsilon)$

We claim that $t \in \int_H^l$ since $ht = h\,\theta^{-1}(\varepsilon) = \theta^{-1}(h \rightharpoonup \varepsilon) = \theta^{-1}(\varepsilon(h)\cdot\varepsilon) = \varepsilon(h)\cdot t$

Note: $\langle h \rightharpoonup \varepsilon, g\rangle = \langle \varepsilon, gh\rangle = \varepsilon(g)\cdot\varepsilon(h) \Rightarrow h \rightharpoonup \varepsilon = \varepsilon(h)\cdot\varepsilon$.

③. H f.d $\overset{\text{mod}}{\curvearrowright} A$, consider A as a right H^*-comodule algebra. Then $\hat{t}$ is surjective $\Leftrightarrow \exists$ a total integral $\varphi: H^* \to A$.

Pf: $\Rightarrow$: $\hat{t}$ is surjective $\Rightarrow \exists c \in A$ s.t. $t\cdot c = 1$

With θ as above, let $\varphi: H^* \to A$
$$f \mapsto \theta^{-1}(f)\cdot c$$

φ is a left H-mod map since θ is, and thus φ is a right H^*-comod map. Moreover, $\varphi(1_{H^*}) = \varphi(\varepsilon) = \theta^{-1}(\varepsilon)\cdot c = t\cdot c = 1$

so φ is a total integral for A.

$\Leftarrow$:

Conversely, assume that $\varphi: H^* \to A$ is a total integral and set $c = \varphi(T)$. Then $t\cdot c = t\cdot\varphi(T) = \varphi(t \rightharpoonup T) = \varphi(\varepsilon) = 1$.

④. dual notion of trace for right H-comod alg A

recall: A is an H-mod alg, $\hat{t}: A \to A^H$ an A^H-bimod. map

Now: A is an right H-comod alg. and $\exists$ right comod map $\varphi: H \to A$.

Setting $tr(a) = a_0\varphi(Sa_1)$, it's easy to check that $tr(a) \in A^{coH}$ and that $tr|_{A^{coH}} = I_{A^{coH}}$ if φ is a total integral.

Cpt 4.4. Ideals in $A\#H$ and A as an A^H-module.

Abstract: How the structure of $A\#H$ influences the relationship between A and A^H

H. f.d., A is a f.g. A^H-module $\hat{t}$ is surjective)

1. Lattice of modules

① Let $e = tc$ and fix a basis $\{h_1, \cdots, h_n\}$ of H. $\forall\, V \in M_A$, let

$W = V \otimes_A (A\#H)$ be the induced $A\#H$-module.

Define $\sigma: \mathcal{L}(V_{A^H}) \longrightarrow \mathcal{L}(W_{A\#H})$, $\mu: \mathcal{L}(W_{A\#H}) \longrightarrow \mathcal{L}(V_{A^H})$

$U \mapsto (U\otimes e)(A\#H)$ $\quad\quad \sum_i v_i \otimes h_i \mapsto \sum_i \varepsilon(h_i) v_i$

lattice of A^H-submodules of V. $\quad\quad$ well-defined.

Note: 1. $\forall\, w \in W$, $w = \sum v_i \otimes a_i h_i = \sum_i v_i a_i \otimes h_i = \sum_i v_i' \otimes h_i$

2. μ is a A^H-mod map. Thus if $X \in \mathcal{L}(W_{A\#H})$, $X^\mu \in \mathcal{L}(V_{A^H})$

pf: $a \in A^H \Rightarrow ha = (h_1 \cdot a) h_2 = a\,\varepsilon(h_1) h_2 = ah$

$\mu(\sum v_i \otimes h_i a) = \mu(\sum_i v_i \otimes a h_i) = \mu(\sum_i v_i a \otimes h_i) = \sum_i \varepsilon(h_i)\, v_i a$

3. both σ and μ preserve inclusion (hence preserve asscending and descending properties) $\rightarrow$ Noetherian.

② $\mu \circ \sigma(U) = U$; henc σ is injective

pf: c: If $v \otimes e = 0$, then $v \otimes et = v \otimes tct = v \otimes t = 0$, so $v = 0$

Hence $v_1 \otimes e = v_2 \otimes e$ implies $v_1 = v_2$.

$\forall\, w = \sum v_i \otimes h_i \in W$, $we = \sum_i v_i \otimes h_i tc = \sum_i \varepsilon(h_i) v_i \otimes tc = \mu(w) \otimes e$.

$\forall\, U \in \mathcal{L}(V_{A^H})$, $U^\sigma e = (U \otimes e)(A\#H)e = U \otimes A^H e = U \otimes e$ by 4.3.4.

$w \in U^\sigma$ implies $\mu(w) \in U$, hence $U^{\sigma\mu} \subseteq U$.

This is true since $we = \mu(w) \otimes e \in U \otimes e$ implies $\mu(w) \in U$

2:

$\forall u \in V$, $w = u \otimes e \in U^{G}$ and $u \otimes e = we = \mu(w) \otimes e$

It follows that $u = \mu(w)$ so $U = U^{G\mu}$.

③. H f.d. $\curvearrowright A$ s.t. $\hat{t}$ is surjective. If A is right Noetherian, then A is a right Noetherian A^H-module.

pf: We apply the lemma with $V = A$, Then $W = A \otimes_A A\#H \cong A\#H$

(7.2.11)

Since $A\#H \cong H \otimes A$ as right A-modules and A is right Noetherian, W is a Noetherian $A\#H$ module.

Now $\underbrace{\mathcal{L}(V_{A^H})}_{\text{a few}} \hookrightarrow \underbrace{\mathcal{L}(W_{A\#H})}_{\text{many}}$, thus A is a Noetherian A^H-mod.

2. Lemma: Let H f.d $\curvearrowright A$ and choose $0 \neq t \in \int_H^{\ell}$, then

1). $ah = h_2(\bar{S}h_1 \cdot a)$ $\quad = t(a \leftharpoonup h)$

2) $hat = (h \cdot a)t$, $tah = t(\bar{S}h^{\alpha} \cdot a)$, $t \cdot h = \alpha(h) \cdot t$, $h^{\alpha} = \alpha \rightharpoonup h$ $\quad \in \int_H^{\ell}$

3) $(t) = AtA$ is an ideal in $A\#H$

pf: (1) $h_2(\bar{S}h_1 \cdot a) = (1\#h_2)(\bar{S}h_1 \cdot a)\#1$

$= (h_2\bar{S}h_1 \cdot a)\#h_3 \cdot 1 = \varepsilon(h_1) \cdot a \# h_2 = ah$

(2). $tah \overset{(1)}{=} t h_2(\bar{S}h_1 \cdot a)$

$= t \cdot \alpha(h_2)(\bar{S}h_1 \cdot a)$

$= t \cdot \bar{S}(\alpha(h_2)h_1 \cdot a)$

$= t \cdot \bar{S}(\alpha \rightharpoonup h \cdot a) = t \cdot \bar{S}(h^{\alpha} \cdot a)$

(3) $\forall a, b \in A$, $h \in H$. $hatb = (h \cdot a)tb \in AtA$

$atbh = at\bar{S}(h^{\alpha} \cdot b) \in AtA$.

Remark: $(t) \downarrow$ influence "A is f.g. A^H-mod ?"

3. consider $(t) = AtA$ in $A\#H$ as above, then

(1). Fix any $a = \sum_{i=1}^{n} b_i t c_i \in (t) \cap A$, then $\forall d \in A$,

$ad = \sum_{i=1}^{n} b_i \hat{t}(c_i d) \in \sum_{i=1}^{n} b_i A^H$

That is $aA \subseteq \sum_{i=1}^{n} b_i A^H$. This says that $I = (t) \cap A$ is "Shirshov locally finite" over A^H.

(2). If $(t) = A \# H$, then A is a f.g. right A^H-module.

(3) If $I = (t) \cap A$ contains a regular element of A, then A is a right A^H-submodule of a finite free A^H-module.

pf: (1). $ad = \sum b_i t \underline{c_i d} = \sum b_i t_1 \cdot (c_i d) \# t_2$. Applying $id \otimes \epsilon$ on both sides, we get $ad = \sum_i b_i (t \cdot c_i d)$

(2). Using $a = 1$ in (1), we get $A \subseteq \sum_{i=1}^{\hat{n}} b_i A^H \subseteq A$ #

(3). Let a be a regular element, i.e. $ad = 0$ implies $d = 0$.

Let $\{b_i\}, \{c_i\}$ be as before, and define

$$\varphi: A \to \bigoplus_{i=1}^{n} A^H, \quad a \mapsto (\hat{t}(c_i d))_i$$, if $\varphi(d) = 0$ then $\hat{t}(c_i d) = 0$, $\forall i$

Thus $ad = \sum_i b_i (t \cdot c_i d) = 0$ and $d = 0$ since a is regular.

Thus φ is injective #.

Note: 1. If $A \# H$ is a simple ring then $(t) \cap A = A$ and hence A is a f.g. A^H-module, and thus A is Noetherian. provided A^H is Noetherian.

4. Semiprime.

① Semiprime ring: The only nilpotent ideal is 0.

②. Thm: Assume that $A \# H$ is semiprime and that every non-zero ideal of $A \# H$ intersects A non-trivially. Then

1). A^H is a (right) Goldie ring $\Leftrightarrow$ A is (right) Goldie.

2) if A^H is (right) Noetherian or Artin, so is A.

③. Lemma: Assume that $A \# H$ is semiprime, and choose $0 \neq t \in \int_H^l$. If I is any non-zero left or right H-stable ideal of A, then $\hat{t}(I) \neq 0$.

pf: If $\hat{t}(I) = 0$, then $tIt = (t \cdot I)t = 0$

Thus if I is a left ideal of A, then $J = It$ is a left ideal of $A \# H$ such that $J^2 = 0$. Since $A \# H$ is semiprime, $J = 0$ and thus $I = 0$, a contradiction. The same argument work for the right part.

Note: J is a left ideal of $A \# H$ since $hJ = hIt = (h \cdot I)t \overset{H\text{-stable}}{\subseteq} It = J$

Remark: 1. 4.4.5. 4.4.6: It's useful to know when $A \# H$ is semiprime.

2. If A is semiprime, $H = \mathbb{K}G$, $|G|^{-1} \in \mathbb{K}$ or $H = (\mathbb{K}G)^*$, then $A \# H$ is semiprime

3. Que: A is semiprime, H is f.d. s.s. $\overset{?}{\Rightarrow}$ $A \# H$ is semiprime.

5. an example: $(t) \cap A = (0)$

recall: $H_4 = \mathbb{K}.\{1, g, x, gx\}$

mul: $g^2 = 1$, $x^2 = 0$, $xg = -gx$

comul: $\Delta g = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$

$\epsilon(g) = 1$, $\epsilon(x) = 0$

$H = H_4$, $A = \mathbb{C}$, $\mathbb{K} = \mathbb{R}$, $B = \mathbb{C} \# H = (t) \oplus (t')$, $t \in \int_H^l$, $t' \in \int_H^r$.

$(t) = BtB = Be_1 \oplus B \cdot e_2$, $e_1 = \frac{1}{2} it$, $e_2 = \frac{1}{2} ti$ $\quad A \cap (t) = (0) = A \cap (t')$

Cpt 4.5. Morita context.

Abstract: 1. $A \# H$ relate A^H via Morita context.

2. Basic idea: relationship via modules. $\xrightarrow{\text{wearker}}$ Morita Equivalence $\swarrow$ module cat. equivalence

4.5.0. definitions.

We say that two rings R and S is connected by Morita context if $\exists M \in {}_R M_S$, $N \in {}_S M_R$ and two bilinear maps

$[\,,\,] : N \otimes_R M \to S$ and $(\,,\,) : M \otimes_S N \to R$

s.t.

(1). $[\,,\,]$ is an S-bimodule map which is middle R-linear.

i.e. $[n\cdot r, m] = [n, r\cdot m]$ (Hence $[\,,\,]$ is well-defined)

(2) $(\,,\,)$ is an R-bimodule map which is middle S-linear.

(3) $\forall\, m, m' \in M,\ n, n' \in N$, "associativity" holds. i.e.

$m' \cdot [n\, m] = (m', n)\cdot m$ and $[m, n]\cdot n' = n \cdot (m, n')$

4.5.1. left, right A^H-mod + left $A\#H$-mod structure.

Let $R = A^H$, $S = A\#H$. and $M = N = A$.

(1). A is a left (or right) A^H-module by multiplication.

(2) A is a left $A\#H$-module in the standard way. i.e.

$(a\#h)\cdot b = a(h\cdot b)$

4.5.2. right $A\#H$-module structure.

Let $\alpha \in H^*$ s.t. $th = \alpha(h)t$, $\forall\, h \in H$, write $h^\alpha = \alpha \rightharpoonup h = \alpha(h_2)\cdot h_1$

By [Radford], $t^\alpha = S t$ for $0 \neq t \in \int_H^l$; in particular, H is unimodular iff $St = t$. $(\alpha = \varepsilon)$. A is a right $A\#H$-module via

$$a \leftarrow b\#h = \bar{S} h^\alpha \cdot (ab)$$

right. right + twisted

4.5.3.

Thm: Let $M = N = A$ be modules defined as above. Then

$M \in {}_{A^H}\mathcal{M}_{A\#H}$ and $N \in {}_{A\#H}\mathcal{M}_{A^H}$ together with the maps

$[\,,\,] : A \otimes_{A^H} A \to A\#H$, $(\,,\,) : A \otimes_{A\#H} A \to A^H$

$a \otimes b \mapsto atb$ $\qquad$ $a \otimes b \mapsto \hat{t}(ab)$

give a Morita context for A^H and $A\#H$.

Pf: We check that $(\,,\,)$ is middle $A\#H$-linear: for $a, b, c \in A$, $h \in H$

$$\begin{aligned}
(c \leftarrow ah, b) &\overset{\text{def of } \leftarrow}{=} (\bar{S} h^{\alpha} \cdot ca, b) && \because (a,b) = \hat{t}(ab) \\
&= \hat{t}((\bar{S}h^{\alpha} \cdot ca) b) && \because \hat{t}(a) = t \cdot a \\
&= t \cdot ((\bar{S}h^{\alpha} \cdot ca) b) && \because h^{\alpha} = \alpha \rightharpoonup h = \alpha(h_2) h_1 \\
&= t\alpha(h_2) \cdot ((\bar{S} h_1 \cdot ca) b) && \because th = \alpha(h) t \\
&= th_2 \cdot ((\bar{S} h_1 \cdot ca) b) && \because h \cdot ab = (h_1 \cdot a)(h_2 \cdot b) \\
&= t \cdot ((h_2 \bar{S} h_1 \cdot ca)(h_3 \cdot b)) && \because h_2 \bar{S} h_1 = \epsilon(h) \\
&= t \cdot (ca(h \cdot b)) && \because \text{left } A\#H\text{-mod } ah \cdot b = a(h \cdot b) \\
&= t \cdot (c(ah \cdot b)) && \because (a,b) = \hat{t}(ab) \\
&= (c, ah \cdot b)
\end{aligned}$$

Remark: $\hat{t}(A) = (A, A)$, $(t) = [A, A]$.

4.5.4.

Let H f.d $\curvearrowright A$, and choose $0 \neq t \in \int_H^{l}$. If both $\hat{t}$ is surjective and $(t) = A\#H$, then $A\#H$ is Morita equivalent to A^H.

4.5.5.

Let H f.d $\curvearrowright A$ s.t. $A\#H$ is a simple ring. Then TFAE:

(1). $A\#H$ is Morita equivalent to A^H.

(2). $\hat{t}$ is surjective

(3). A^H is simple.

(1) $\Leftrightarrow$ (2)

pf: Since $A\#H$ is simple, $(t) = A\#H$ and so $[\,,\,]$ is surjective.

Thus, by 4.5.4. $A\#H$ is Morita equivalent to $A^H \Leftrightarrow \hat{t}$ is surjective

(1) $\Rightarrow$ (3)

A^H is simple since being simple is a Morita invariant.

(3) $\Rightarrow$ (1)

Since $\hat{t}(A)$ is an ideal of A^H and is non-zero by 4.4.6.

Thus $\hat{t}(A) = A^H$ if A^H is simple, and so $\hat{t}$ is surjective.

4.5.6. prime ring

① A ring is prime if the product of non-zero ideals is non-zero.

② Let H f.d. $\curvearrowright A$, then $A\#H$ is a prime ring $\Leftrightarrow$ A is a left and right faithful $A\#H$-module and A^H is a prime ring.

Remark: example 4.4.8 shows that A^H and $A\#H$ are not always Morita equivalent

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Summary. 4.3–4.5.

Cpt 4.3.

1. $\hat{t}: A \to A^H$ is an A^H-bimodule.

   $a \mapsto t\cdot a$

2. $\hat{t}$ is surjective iff $\exists c \in A$ s.t. $t\cdot c = 1$. In this case, $e = tc$ is a non-zero idempotent s.t. $e(A\#H)e = A^H e \cong A^H$

3. let H f.d. $\curvearrowright A$, $\hat{t} \twoheadrightarrow$

   A is left/right Noetherian $\Rightarrow$ so is A^H

   A is left Noetherian, $\mathbb{k}$-affine $\Rightarrow$ A^H is $\mathbb{k}$-affine.

   (4.2.5: H f.d. cocomm. A comm. $\Rightarrow$ A^H is $\mathbb{k}$-affine)

4. total integral: $H \overset{\text{comod}}{\curvearrowright} A$, $\varphi: H \to A$ right H-mod s.t. $\varphi(1) = 1$.

   $H \overset{\text{mod}}{\curvearrowright} A$ s.t. $\hat{t}$ surjective $\Leftrightarrow$ $H^* \overset{\text{comod}}{\curvearrowright} A$ $\exists$ total integral φ.

5. dual notion: $H \overset{\text{comod}}{\curvearrowright} A$, $\exists\, \varphi: H \to A$ right H-comod. $\Rightarrow$ $tr(a) = a_0 \varphi(S a_1)$

Cpt 4.4.

1. Lattice of modules

   let H f.d $\curvearrowright A$, $\hat{t} \twoheadrightarrow$, $V \in M_A$, $W = V \otimes_A A\#H \in M_{A\#H}$, then

   $\exists$ $\sigma: \mathcal{L}(V_{A^H}) \to \mathcal{L}(W_{A\#H})$, $\mu: \mathcal{L}(W_{A\#H}) \to \mathcal{L}(A^H)$

   s.t. $\mu \circ \sigma\, U = U$, hence σ is injective

(2) A is right Noetherian $\Rightarrow$ A is a right Noetherian A^H-module.

2. H f.d $\curvearrowright A$, $0 \neq t \in \int_H^l$ (t is not necessarily surjective), then

(1) $ah = h_2(\bar{S}h_1 \cdot a)$

(2) $hat = (h \cdot a)t$, $tah = t(a \leftharpoonup h)$. where $a \leftharpoonup h = \bar{S}h^\alpha \cdot a$

(3) $(t) = AtA$ is an ideal in $A\#H$

3. (1) $\forall a \in A \cap (t)$, $\exists \{b_i\}$ s.t. $aA \subseteq \sum_{i=1}^n b_i A^H$

(2) $(t) = A\#H \Rightarrow A \cap (t) = A$

$\Rightarrow A \subseteq \sum_{i=1}^n b_i A^H$, is a f.g. A^H-module.

(3). $(t) \cap A$ contains a regular element $\Rightarrow A \hookrightarrow (A^H)^{(n)}$, $\exists n$ as right A^H-module.

4. Property of semiprime; example of $(t) \cap A = 0$.

Cpt 4.5. Morita context.

1. def: R, S is conneted by Morita context if $\exists M, N$ with $[,]$, $(,)$ s.t. (1)–(3).

2. $A\#H$ and A is conneted by Morita context via A, with $[,]$, $(,)$.

3. $\hat{t}(A) = [A, A]$, $(t) = (A, A)$.

$A\#H$ is Morita equivalent to A^H if $[,]$ and $(,)$ are surjective.

4. Morita equivalent & simple ring.