Cpt1. Def & Examples.

Cpt1.1 alg & coalg

1. Hopf: $(H, \Delta, \epsilon, m, u)$

① $\begin{cases} (A, m, u): \text{algebra} \\ (C, \Delta, \epsilon): \text{coalgebra} \end{cases} \xrightarrow[\text{condiction}]{\text{compatible}}$ bialgebra $\xrightarrow{\text{antipode}}$ Hopf alg.

② $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\qquad 1\cdot a = a = a\cdot 1$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "1_A\otimes m"] \arrow[d, "m\otimes 1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} & A\otimes A \arrow[d, "m"] & \\ \mathbb{K}\otimes A \arrow[ur, "u\otimes 1_A"] \arrow[r, phantom, "\circlearrowright"] \arrow[r, phantom, shift right=3, "\cong"'] & A & A\otimes \mathbb{K} \arrow[ul, "1_A\otimes u"'] \arrow[l, phantom, "\circlearrowleft"] \arrow[l, phantom, shift left=3, "\cong"] \end{tikzcd}$$

associativity $\qquad\qquad\qquad$ unit

③
$$\begin{tikzcd} C\otimes C\otimes C \arrow[dr, phantom, "\circlearrowleft"] & C\otimes C \arrow[l, "I_C\otimes\Delta"'] \\ C\otimes C \arrow[u, "\Delta\otimes 1_C"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & C\otimes C \arrow[dl, "\epsilon\otimes 1_C"'] \arrow[dr, "1_C\otimes\epsilon"] & \\ \mathbb{K}\otimes C \arrow[r, phantom, "\circlearrowright"] \arrow[r, phantom, shift right=3, "\cong"'] & C \arrow[u, "\Delta"] & C\otimes\mathbb{K} \arrow[l, phantom, "\circlearrowleft"] \arrow[l, phantom, shift left=3, "\cong"] \end{tikzcd}$$

coassociativity $\qquad\qquad\qquad$ counit

④ Example

grp. alg: $C = \mathbb{K}G = \mathbb{K}.\{g\in G\}$

$$\Delta(g) = g\otimes g, \quad \epsilon(g) = 1$$

Note: Structure = basis + map

Pf:

$$g \xrightarrow{\Delta} g\otimes g \xrightarrow{1_C\otimes\Delta} g\otimes(g\otimes g)$$
$$\parallel$$
$$g \xrightarrow{\Delta} g\otimes g \xrightarrow{\Delta\otimes 1_C} (g\otimes g)\otimes g$$

$$\begin{tikzcd} & g\otimes g \arrow[dl, "\epsilon\otimes 1_C"'] \arrow[dr, "I_C\otimes\epsilon"] & \\ 1\otimes g \arrow[r, phantom, "="] & g \arrow[u, "\Delta"] & g\otimes 1 \arrow[l, phantom, "="] \end{tikzcd}$$

Ex: $C = \mathbb{K}[x]$, $\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k}\cdot x^k\otimes x^{n-k}$

$$\epsilon(x^n) = \delta_{n0}$$

(C, Δ, ϵ) is a coalgebra.

2. homomorphism

① $f: A \to B$ is a alg hom:

$$\begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "m_A"'] & B\otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & k \arrow[dl, "u_A"'] \arrow[dr, "u_B"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

$f(a\cdot b) = f(a)\cdot f(b)$ $\qquad$ $f(u_A(1)) = u_B(1)$

② $g: C \to D$ is a coalg hom:

$$\begin{tikzcd} D\otimes D & C\otimes C \arrow[l, "g\otimes g"'] \\ D \arrow[u, "\Delta_D"] & C \arrow[l, "g"] \arrow[u, "\Delta_C"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & k & \\ D \arrow[ur, "\varepsilon_D"] & & C \arrow[ll, "g"] \arrow[ul, "\varepsilon_C"'] \end{tikzcd}$$

③ $I \lhd C$: $\Delta(I) \subseteq I\otimes C + C\otimes I$ and $\varepsilon(I) = 0$

$\iff \exists f$ be a coalg. hom s.t. $I = \mathrm{Ker} f$

1st. iso. thm : $C/\mathrm{Ker} f \cong \mathrm{Im} f$

Remark: In general : $M' \subseteq M,\ N' \subseteq N \not\Rightarrow M' \otimes_R N' \hookrightarrow M \otimes N$

Example: $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$

However, $\mathbb{K}$ is a field, $\Rightarrow V' \otimes_{\mathbb{K}} V' \hookrightarrow V \otimes V$ (pf omitted)

3. others : twist (flip) τ

opposite : op, cop

commutative, cocommutative

Note: Coideal 从作用上，类比 ideal，但对偶性质上类比 sub alg.

Cpt 1.2. Dual of coalg and alg.

1. transpose

① $\langle , \rangle : V^* \otimes V \to \mathbb{K}$, ($V^* = \mathrm{Hom}(V, \mathbb{K})$)

$v^* \otimes v \mapsto v^*(v)$

i.e. $\langle v^*, v \rangle := v^*(v)$

② $f: V \to W \in \mathrm{Hom}(V,W)$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "v^*"'] & W \arrow[d, "w^*"] \\ & \mathbb{K} \end{tikzcd}$$

$\Rightarrow {}^t f: W^* \to V^* \in \mathrm{Hom}(W^*, V^*)$
$\quad \; w^* \mapsto w^* \circ f$

$f^* = {}^t f$

i.e. $t: \mathrm{Hom}(V,W) \to \mathrm{Hom}(W^*, V^*) \longrightarrow \mathrm{Hom}(V^{**}, W^{**})$

$$f \mapsto f^*: \begin{pmatrix} W^* \to V^* \\ w^* \mapsto w^* \circ f \end{pmatrix}$$

③ $\langle {}^t f \cdot w^*, v \rangle = \langle w^*, f(v) \rangle$

Note: t 为嵌入，使用 $\langle , \rangle$ 在处理对偶时，记号不易乱.

2. Lemma:
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "h"'] & V \arrow[d, "g"] \\ & W \end{tikzcd}$$
is commutative

$\Leftrightarrow$
$$\begin{tikzcd} U^* & V^* \arrow[l, "f^*"'] \\ & W^* \arrow[u, "g^*"'] \arrow[ul, "h^*"] \end{tikzcd}$$
is commutative

pf: $\Rightarrow$: $\because \langle f^* \cdot g^* \cdot w^*, u \rangle = \langle g^* \cdot w^*, f(u) \rangle \quad \forall u \in U$

$= \langle w^*, g \circ f(u) \rangle$

$= \langle (g \circ f)^* \cdot w^*, u \rangle$

$\therefore \; f^* \cdot g^* = (g \circ f)^*$

$\Leftarrow$: t is injective.

3.① coalg $\xrightarrow{\text{dual}}$ alg

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "I_C \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes I_C"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}
\qquad
\begin{tikzcd} & C \otimes C \arrow[dl, "\epsilon \otimes 1_C"'] \arrow[dr, "1_C \otimes \epsilon"] & \\ \mathbb{K} \otimes C & & C \otimes \mathbb{K} \\ & C \arrow[ul, equal] \arrow[uu, "\Delta"] \arrow[ur, equal] & \end{tikzcd}$$

$\Downarrow$

$$\begin{tikzcd} (C \otimes C \otimes C)^* \arrow[r, "(I_C \otimes \Delta)^*"] \arrow[d, "(\Delta \otimes 1_C)^*"'] & (C \otimes C)^* \arrow[d, "\Delta^*"] \\ (C \otimes C)^* \arrow[r, "\Delta^*"] & C^* \end{tikzcd}
\quad \& \quad
\begin{tikzcd} & (C \otimes C)^* \arrow[dd, "\Delta^*"] & \\ \mathbb{K}^* \otimes C^* \arrow[ur, "(\epsilon \otimes 1_C)^*"] & & C^* \otimes \mathbb{K}^* \arrow[ul, "(1_C \otimes \epsilon)^*"'] \\ & C^* \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
C^*\otimes C^*\otimes C^* \arrow[r, "1_{C^*}\otimes\Delta^*"] \arrow[d, "\Delta^*\otimes 1_{C^*}"'] & C^*\otimes C^* \arrow[d, "\Delta^*"] \\
C^*\otimes C^* \arrow[r, "\Delta^*"'] & C^*
\end{tikzcd}
\quad \&\quad
\begin{tikzcd}
 & C^*\otimes C^* \arrow[d, "\Delta^*"] & \\
\mathbb{K}\otimes C^* \arrow[ur, "\epsilon^*\otimes 1_{C^*}"] \arrow[r, phantom, "\circlearrowright"] & C^* \arrow[l, phantom, "\cong"] \arrow[r, phantom, "\cong"] & C^*\otimes\mathbb{K} \arrow[ul, "1_{C^*}\otimes\epsilon^*"'] \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}$$

where : $C^*\otimes C^* \hookrightarrow (C\otimes C)^*$

$$u^*\otimes v^* \mapsto \left(\begin{array}{l} C\otimes C\to\mathbb{K} \\ u\otimes v\mapsto u^*(u)\cdot v^*(v)\end{array}\right)$$

then $(C^*, m, u) = (C^*, \Delta^*|_{C^*\otimes C^*}, \epsilon^*)$ is an alg.

② $m: C^*\otimes C^*\to C^*$

$$c^*\otimes d^* \mapsto m(c^*\otimes d^*)$$

$$\langle m(c^*\otimes d^*), c\rangle = \langle \Delta^*(c^*\otimes d^*), c\rangle$$

$$= \langle c^*\otimes d^*, \Delta(c)\rangle$$

Note: $\mathbb{K}G$ 中, $(c^*\otimes d^*)\circ\Delta(g) = (c^*\otimes d^*)(g\otimes g)$, $\forall g\in G$

$$= c^*(g)\cdot d^*(g)$$

(线性)

$$\Rightarrow (c^*\otimes d^*)\circ\Delta(2\cdot g) = (c^*\otimes d^*)(2\cdot g\otimes g)$$

$$= 2\cdot c^*(g)\cdot d^*(g)$$

$$\neq (c^*\otimes d^*)(2g\otimes 2g)$$

③ alg $\longrightarrow$ coalg .

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "1_A\otimes m"] \arrow[d, "m\otimes 1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\iff
\begin{tikzcd}
(A\otimes A\otimes A)^* & (A\otimes A)^* \arrow[l, "(1_A\otimes m)^*"'] \\
(A\otimes A)^* \arrow[u, "(m\otimes 1_A)^*"] \arrow[ur, phantom, "\circlearrowleft"] & A^* \arrow[l, "m^*"] \arrow[u, "m^*"']
\end{tikzcd}$$

Remark : (1) $m^*(A^*)$ 未必在 $A^*\otimes A^*$ 中，令 $A^\circ = m^{*-1}(A^*\otimes A^*)\subseteq A^*$

使 A° 作成余代数且在 A^* 中相大.

(2) $A^\circ = \{f\in A^* \mid \exists\, I$ be a cofinite ideal of A s.t. $f(I)=0\}$

Cpt 1.3. Bialgebras

1. A vec. space B is a bialgebra.

if $\begin{cases} (B,\Delta,\epsilon) \text{ is a coalg} \\ (B,m,u) \text{ is an alg} \end{cases}$ and. Δ and ϵ are alg. homs. or m and u are coalg homs

2. A, B 为代数 $\Rightarrow$ $A\oplus B$, $A\otimes B$ 为代数

具体地, (1) $m: (A\oplus B)\otimes(A\oplus B) \to A\oplus B$ $\quad u: \mathbb{K} \to A\oplus B$

$(a_1,b_1)\otimes(a_2,b_2) \mapsto (a_1a_2, b_1b_2)$ $\quad 1 \mapsto (1,1)$

(2) $m: (A\otimes B)\otimes(A\otimes B) \to A\otimes B$ $\quad u: \mathbb{K} \to A\otimes B$

$(a_1\otimes b_1)\otimes(a_2\otimes b_2) \mapsto (a_1\cdot a_2)\otimes(b_1\cdot b_2)$ $\quad 1 \mapsto 1\otimes 1$

C, D 为余代数 $\Rightarrow$ $C\oplus D$, $C\otimes D$ 作成余代数.

具体地, (1) $\Delta: C\otimes D \to (C\otimes D)\otimes(C\otimes D)$, $\epsilon: C\otimes D \to \mathbb{K}$

$c\otimes d \mapsto (c_{(1)}\otimes d_{(1)})\otimes(c_{(2)}\otimes d_{(2)})$ $\quad c\otimes d \mapsto \epsilon(c)\cdot\epsilon(d)$

(2) $\Delta: C\oplus D \to (C\oplus D)\otimes(C\oplus D)$, $\epsilon: C\oplus D \to \mathbb{K}$

$(c,d) \mapsto (c_{(1)},d_{(1)})\otimes(c_{(2)},d_{(2)})$ $\quad (c,d) \mapsto \epsilon(c)+\epsilon(d)$

3. Δ, ϵ are alg. homs $\Leftrightarrow$ m, u are coalg homs

pf: (1) $\Delta: B \to B\otimes B$. is an alg. hom $\Leftrightarrow$

$$\begin{tikzcd} B\otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"'] & (B\otimes B)\otimes(B\otimes B) \arrow[d, "m_{B\otimes B}"] \\ B \arrow[r, "\Delta"] & B\otimes B \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbb{K} \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\ B \arrow[rr, "\Delta"] & & B\otimes B \end{tikzcd}$$

$\epsilon: B \to \mathbb{K}$ is an alg. hom $\Leftrightarrow$

$$\begin{tikzcd} B\otimes B \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & \mathbb{K}\otimes\mathbb{K} \arrow[d, "m_{\mathbb{K}\otimes\mathbb{K}}"] \\ B \arrow[r, "\epsilon"] & \mathbb{K} \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbb{K} \arrow[dl, "u"'] \arrow[dr, "u_{\mathbb{K}}"] & \\ B \arrow[rr, "\epsilon"] & & \mathbb{K} \end{tikzcd}$$

$\Leftrightarrow$

$$\begin{tikzcd} B\otimes B \arrow[d, "m"'] \arrow[dr, "\epsilon\otimes\epsilon"] & \\ B \arrow[r, "\epsilon"] & \mathbb{K} \end{tikzcd}$$

and $\epsilon\circ u = 1_{\mathbb{K}}$

$B \longrightarrow \mathbb{K}$

(2)

$m: B\otimes B \to B$ is a coalg hom $\iff$

$$\begin{tikzcd} B\otimes B \arrow[r, "\Delta_{B\otimes B}"] \arrow[d, "m"] & (B\otimes B)\otimes(B\otimes B) \arrow[d, "m\otimes m"] \\ B \arrow[r, "\Delta"] & B\otimes B \end{tikzcd}$$

and.

$$\begin{tikzcd} & \mathbb{K} & \\ B\otimes B \arrow[ur, "\varepsilon\otimes\varepsilon"] \arrow[rr, "m"] & & B \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

Similarly, $u: \mathbb{K} \to B$ is a coalg hom $\iff$

$$\begin{tikzcd} & \mathbb{K} \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\ B \arrow[rr, "\Delta"] & & B\otimes B \end{tikzcd}$$

and $\varepsilon\circ u = 1_{\mathbb{K}}$

Noticed that: the first diagram of m and Δ is equivalent to

$$\boxed{\begin{tikzcd} & B \arrow[dr, "\Delta"] & \\ B\otimes B \arrow[ur, "m"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[dr, "\Delta_{B\otimes B}"] & & B\otimes B \\ (B\otimes B)\otimes(B\otimes B) \arrow[r, "1_B\otimes\tau\otimes 1_B", "\sim"'] & B\otimes B\otimes B\otimes B \arrow[ur, "m\otimes m"'] \arrow[ul, "m_{B\otimes B}"] & \end{tikzcd}}$$

Remark: Equivalent condition (straight from the proof)

(1) $\varepsilon\circ u = 1_{\mathbb{K}}$, i.e. $\overset{(\text{unit})}{\varepsilon(1)} = 1$

(2) $\varepsilon\circ m = \varepsilon\otimes\varepsilon$, i.e. $\varepsilon(a\cdot b) = \varepsilon(a)\cdot\varepsilon(b)$

(3) $\Delta\circ u = u\otimes u$, i.e. $\Delta(1) = 1\otimes 1$

(4) $\Delta\circ m = m_{B\otimes B}\circ \Delta\otimes\Delta$, i.e. $c_{(1)}d_{(1)}\otimes c_{(2)}d_{(2)} = (cd)_{(1)}\otimes(cd)_{(2)}$

$\qquad\qquad\qquad\quad \Updownarrow$

$\qquad\qquad m\otimes m\circ\Delta_{B\otimes B}$

4. 1st. hom. thm.

① bialgebra map: both an alg. and coalg. morphism.

biideal: both an alg. and coalg. ideal.

② $I \lhd B \iff \exists f: B\to B'$ bialg. map. s.t. $I = \operatorname{Ker} f$

$B/\operatorname{Ker} f \cong \operatorname{Im} f$.

5. Examples:

① grp alg. $\mathbb{K}G$. ($\Delta(g) = g\otimes g$)

②. let $\mathfrak{g}$ be a $\mathbb{K}$-Lie alg. $B = U(\mathfrak{g})$ be its universal Enveloping algebra. let $\Delta(x) = 1\otimes x + x\otimes 1$, $\epsilon(x) = 0$,

both $\mathbb{K}.G$ and $U(\mathfrak{g})$ are cocommutative.

6. ①. grouplike element: $g \in C$ s.t. $\Delta(g) = g\otimes g$ and $\epsilon(g) = 1$

$G(C) = \{g \in C \mid g$ is a grouplike element$\}$ is a subco alg.

②. For $g, h \in G(C)$, $c \in C$ is called g,h-primitive element if $\Delta(c) = c\otimes g + h\otimes c$

$P_{g,h}(C) = \{c \in C \mid c$ is g,h-primitive$\}$

Note: $c \in P_{g,h}(c) \Rightarrow \epsilon(c) = \epsilon\otimes\epsilon \circ \Delta(c) = \epsilon(c)\cdot\epsilon(g) + \epsilon(h)\cdot\epsilon(c) = 2\epsilon(c)$

$\Rightarrow \epsilon(c) = 0$.

$P_{g,h}(C) \trianglelefteq C$, Since $\Delta(P_{g,h}(c)) \subseteq P_{g,h}(c)\otimes C + C\otimes P_{g,h}(C)$,

In fact, any subspace of $P_{g,h}(c)$ is an coideal of C.

③ B is a bialgebra, $P(B) := P_{1,1}(B)$

Elements of $P(B)$ are called primitive.

7. $B = \mathbb{K}.G \Rightarrow G(B) = G$

$B = U(\mathfrak{g})$, $\operatorname{char}\mathbb{K} = 0 \Rightarrow P(B) = \mathfrak{g}$

$B = U(\mathfrak{g})$, $\operatorname{char}\mathbb{K} = p \Rightarrow P(B) = \operatorname{span}\{x^{p^k} \mid k \geq 0, x \in \mathfrak{g}\}$

which is a restricted p-lie alg.

let A be an algebra

define $\operatorname{Alg}(A, \mathbb{K}) = \{f \in A^* \mid f$ is an alg. map$\}$

then. $\operatorname{Alg}(A, \mathbb{K}) = G(A^\circ)$ (Cpt 9)

8. B is an bialg $\Rightarrow$ B° is an bialg. (Cpt 4)

B is bialgebra $\Rightarrow$ B^{op}, B^{cop}, $B^{op,cop}$ are bialgebras

Cpt1.4. Convolution and sweedler notation.

By default, C is a coalgebra and A is an algebra.

1. convolution (卷积)

① $(\mathrm{Hom}_k(C,A), m, u)$ is an algebra.

where : $m = *: \mathrm{Hom}_k(C,A) \otimes \mathrm{Hom}_k(C,A) \longrightarrow \mathrm{Hom}_k(C,A)$

$$f \otimes g \longmapsto m \circ (f \otimes g) \circ \Delta$$

$*$ is called the convolution product

$u \circ \varepsilon$ is its unit.

② $C^* = \mathrm{Hom}(C, \mathbb{K})$ is an algebra.

$f, g \in C^* \Rightarrow f * g(c) = m \circ (f \otimes g) \circ \Delta(c)$

③ anti-convolution

$\times : \mathrm{Hom}_k(C,A) \otimes \mathrm{Hom}_k(C,A) \longrightarrow \mathrm{Hom}_k(C,A)$

$$f \otimes g \longmapsto m \circ (f \otimes g) \circ \tau \circ \Delta$$

2. sweedler notation.

① $\Delta : C \to C \otimes C \Rightarrow \Delta(c) = \sum_{i=1}^{r} u_i \otimes v_i$

write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, for short and "$\Sigma$" can be omitted.

② coassociativity

$(I_C \otimes \Delta) \circ \Delta(c) = (\Delta \otimes I_C) \circ \Delta(c)$

$\Leftrightarrow (I_C \otimes \Delta)(c_{(1)} \otimes c_{(2)}) = (\Delta \otimes I_C)(c_{(1)} \otimes c_{(2)})$

$\Leftrightarrow c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$

write $\Delta_2 = (I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta$

then $\Delta_2(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ is well defined by coasso.

③ $\Delta^i = I_C \otimes \cdots \otimes \overset{(i)}{\Delta} \otimes \cdots \otimes I_C$ (i items in all)

$\Delta_n = \Delta^n \otimes \cdots \otimes \Delta^1$ is independent of the choice of $\{j\}$.

write $\Delta_{n-1}(C) = c_{(1)} \otimes \cdots \otimes c_{(n)}$ for short.

3. Example: $c_{(1)} \otimes c_{(2)(2)} \otimes c_{(2)(1)} \otimes c_{(3)(2)(1)} \otimes c_{(3)(2)(2)} \otimes c_{(3)(1)}$

4. property: define $\epsilon^{i} = I_C \otimes \cdots \otimes \overset{(i)}{\epsilon} \otimes \cdots \otimes I_C$ (i items in all)

then $\epsilon^{n+1} \circ \Delta_n = \Delta_{n-1}$

pf: unit: $(\epsilon \otimes I_C) \circ \Delta = I_C = (I_C \otimes \epsilon) \circ \Delta$

$\Leftrightarrow \quad \epsilon(c_{(1)}) \cdot c_{(2)} = c_{(1)} \cdot \epsilon(c_{(2)})$

5. $(f * g)(c) = m \circ (f \otimes g) \circ \Delta(c) = f(c_{(1)}) \cdot g(c_{(2)})$

Remark: 引入 sweedler notation 便于推导及体现式子对称性

Cpt 1.5. Antipode and Hopf alg.

1. ① Hopf alg. $\begin{cases} (H, m, u, \Delta, \epsilon) \text{ is an bialgebra} \\ \exists\, S \in \mathrm{Hom}_{\mathbb{K}}(H,H) \text{ s.t. } S * 1_H = 1_H * S = u \circ \varepsilon \end{cases}$

② diagram

$$\begin{tikzcd} & H \otimes H \arrow[r, "S \otimes I_H"] & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbb{K} \arrow[rr, "u"] & & H \\ & H \otimes H \arrow[r, "I_H \otimes S"] & H \otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

③ sweedler: $S(h_{(1)})h_{(2)} = \epsilon(h) \cdot 1_H = h_{(1)} \cdot S(h_{(2)})$

1_H 与 I_H 前面弄错了!

2. $f: H \to K$ is called a Hopf morphism

if it is a bialg. hom. and $f(S_H h) = S_K f(h)$

i.e.
$$\begin{tikzcd} H \arrow[r, "f"] \arrow[d, "S_H"'] & K \arrow[d, "S_K"] \\ H \arrow[r, "f"] & K \end{tikzcd}$$

I is a Hopf ideal if I is a biideal and $SI \subseteq I$

3. Examples.

① $H = \mathbb{K}.G$:
$$\begin{tikzcd} & g \otimes g \arrow[r] & S(g) \cdot g \arrow[d, phantom, "\shortparallel"] \\ g \arrow[ur] \arrow[r, "\varepsilon"] \arrow[dr] & 1 \arrow[r, "u"] & 1_H \\ & g \otimes g \arrow[r] & g \cdot S(g) \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

. Corollary: $g \in G(H)$

$\Rightarrow S_H(g) = g^{-1}$

②. $H=U(g)$

$$\begin{array}{ccccc} & \nearrow & g\otimes 1+1\otimes g & \to & S(g)+g \\ g & \xrightarrow{\epsilon} & 0 & \xrightarrow{u} & 0 \\ & \searrow & g\otimes 1+1\otimes g & \to & g+S(g) \end{array}$$

Corollary: $g\in P(H)$ $\Rightarrow$ $S_H(g)=-g$

③ H is a Hopf alg $\Rightarrow$ H° is also a Hopf alg. (Cpt 9)
with antipode S^*

④. $H_4 = \mathbb{k}.\{1, g, x, gx\}$
mul : $g^2=1,\ x^2=0,\ xg=-gx$
comul : $\Delta g = g\otimes g$, $\Delta x = x\otimes 1+g\otimes x$
$\epsilon(g)=1$, $\epsilon(x)=0$

Ex: determine S_H and find its order

Remark: $o(S_H)=2n$ or infinite.

⑤. $B=O(M_n(\mathbb{k}))=\mathbb{k}[x_{ij}\mid 1\le i,j\le n]=\mathbb{k}.\{x_{i_1j_1}^{k_1}\cdots\cdots x_{i_rj_r}^{k_r}\mid k_i\in\mathbb{Z}_+\}$

mul : polynomial ring

comul: $\Delta(x_{ij})=\sum_{k=1}^{n} x_{ik}\otimes x_{kj}$, $\epsilon(x_{ij})=\delta_{ij}$

let $X=(x_{ij})_{n\times n}$, then $\det X\in G(B)$ ($n=1$ ✓, $n=2$, $\cdots\cdots$)

$\det X=\sum_{\sigma\in S_n}(-1)^{\operatorname{sgn}\sigma}\cdot x_{1\sigma(1)}\cdot\cdots\cdots x_{n\sigma(n)}$ is inreversible

contradictory to "$g\in G(H)\Rightarrow S_H(g)=g^{-1}$".

closely hopf algs related to B:

- $O(M_n(\mathbb{k}))/(\det X-1)$
- $O(M_n(\mathbb{k}))[\det X^{-1}]$

4. Prop: S is both anti-alg. and anti-coalg. (Sweedler Cpt4)

i.e. $S\circ m = m\circ(S\otimes S)\circ\tau$

$\Delta\circ S=\tau\circ(S\otimes S)\circ\Delta$

or: $S(h\cdot k)=S(k)\cdot S(h)$, $S(1)=1$

$S(h_{(2)})\otimes S(h_{(1)}) = S(h)_{(1)}\otimes S(h)_{(2)}$

5. $(H, S) \Rightarrow (H^{op,cop}, \overline{S})$.

where $\overline{S}$ is the inverse of 1_H under twisted convolution.

$(f * g)(c) = f(c_{(1)}) \cdot g(c_{(2)})$, $(f \times g)(c) = f(c_{(2)}) \cdot g(c_{(1)})$

$\overline{S}(h_{(2)}) \cdot h_{(1)} = h_{(2)} \cdot \overline{S}(h_{(1)}) = \varepsilon(h) \cdot 1_H$

6. B is a bialg, then B is a Hopf alg with (composition) invertible antipode $S \Leftrightarrow B^{cop}$ is ...... $\overline{S}$

in this case, $\overline{S} = S^{-1}$ (i.e. both $\circ$ and $*$)

Pf: (omitted, 先自己证!)

Corollary: H is cocomu. or commu $\Rightarrow S^2 = Id$.

Remark: $S^2 \neq id$ 相关问题 Cpt 2 中讨论

以上，Hopf 基本定义: $\left\{\begin{matrix} \text{alg} \\ \text{coalg} \end{matrix}\right.$ + $\begin{matrix} m, u \\ \text{or} \\ \Delta, \varepsilon \end{matrix}$ + antipode.

附: Example: $A^\circ = A^*$

$A = \mathbb{k}.\{a_i \mid i \in I\}$ with $1 = a_0 \in I$

$m: A \otimes A \to A$ $\qquad\qquad$ $\varepsilon: \mathbb{k} \to A$

$a_i \otimes a_j \mapsto \begin{cases} 1, & i=j=0 \\ 0, & \text{other cases} \end{cases}$ $\qquad\qquad$ $1 \mapsto a_0$

let $I = \text{span}\{a_i \mid i \neq 0\}$ be an cofinite ideal of A

then any subspace of I is also an ideal of A

$\forall f \in A^*$, $\text{Ker} f$ is cofinite $\Rightarrow I \cap \text{Ker} f$ is a cofinite ideal of A

$\Rightarrow f \in A^\circ$ #

In fact. $\Delta = m^*: A^* \to (A \otimes A)^*$

$a^* \mapsto \langle \Delta(a^*), - \rangle$

we prove that $\text{Im}\,\Delta \subseteq A^* \otimes A^*$.

i.e. $\langle \Delta(a^*), - \rangle = \langle a^*_{(1)} \otimes a^*_{(2)}, - \rangle$

$$\langle \Delta(a^*), u\otimes v\rangle = \langle m^*(a^*), u\otimes v\rangle \quad , u, v \in \text{Basis of } A.$$

$$= \langle a^*, m(u\otimes v)\rangle$$

$$= \langle a^*, u\cdot v\rangle$$

$$= \begin{cases} a^*(1), & u=v=1 \\ a^*(u), & u\neq 1, v=1 \\ a^*(v), & u=1, v\neq 1 \\ 0 & u\neq 1\neq v \end{cases}$$

$$= \left(\begin{array}{l} a^*(u)\cdot 1^*(v) + 1^*(u)\cdot a^*(v) \\ -a^*(1)\cdot 1^*(u)\cdot 1^*(v) \end{array} \right)$$

$$= \langle a^*\otimes 1^* + 1^*\otimes a^* - a^*(1)\cdot 1^*\otimes 1^*, u\otimes v\rangle$$

hence $\Delta: A^* \longrightarrow A^*\otimes A^*$

$$a^* \mapsto a^*\otimes 1^* + 1^*\otimes a^* - a^*(1)\cdot 1^*\otimes 1^*$$

Example. $A^\circ = 0$.

let $A = \mathbb{K}(t)$ be an infinit-dimensional alg.

then $I \trianglelefteq A \Rightarrow I = 0$ or A

$\Rightarrow A^\circ = 0$