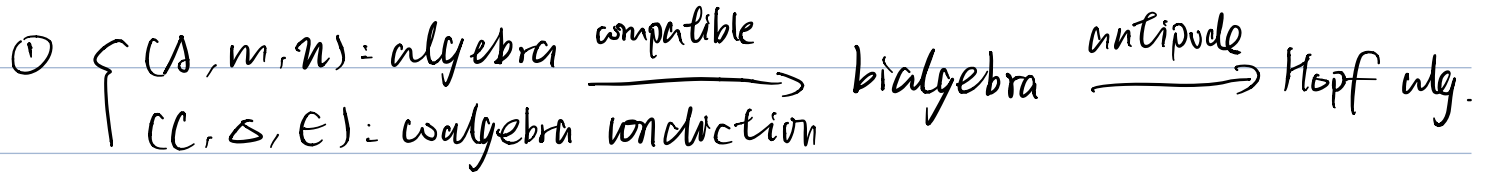


Cptl. Def & Examples.

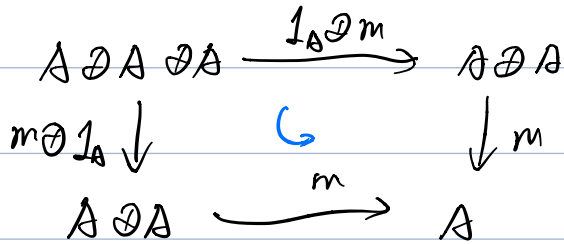
Cptl. alg & walg

1. Hopf: $(H, \Delta, \epsilon, m, \eta)$

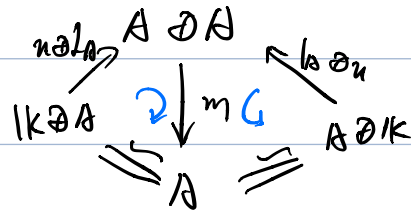


② $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

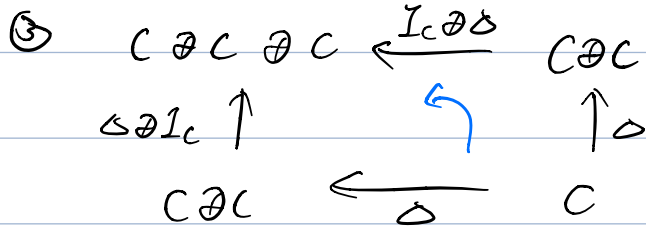
$1 \cdot a = a = a \cdot 1$



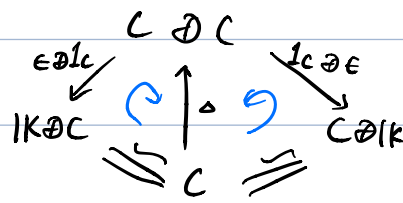
associativity



unit



coassociativity



counit

④ Example

grp. alg: $C = \mathbb{K}G = \mathbb{K} \cdot \{g \in G\}$

$\Delta(g) = g \otimes g, \epsilon(g) = 1$

Note: Structure = basis + map

pf:

$g \xrightarrow{\Delta} g \otimes g \xrightarrow{I_C \otimes \Delta} g \otimes (g \otimes g)$
 $\Delta \otimes I_C \uparrow \quad \quad \quad \uparrow \Delta$
 $g \otimes g \xrightarrow{\Delta} (g \otimes g) \otimes g$

$\epsilon \otimes I_C \downarrow \quad \quad \quad \downarrow I_C \otimes \epsilon$
 $I_C \otimes g \uparrow \quad \quad \quad \uparrow I_C \otimes g$
 $g \otimes g \xrightarrow{\Delta} g \otimes g$

Ex: $C = \mathbb{K}[x], \Delta(x^n) = \sum_{k=0}^n \binom{n}{k} \cdot x^k \otimes x^{n-k}$

$\epsilon(x^n) = \delta_{n0}$

(C, Δ, ϵ) is a coalgebra.

2. homomorphism

① $f: A \rightarrow B$ is a alg hom:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ m_A \downarrow & & \downarrow m_B \\ A & \xrightarrow{f} & B \end{array}$$

and

$$\begin{array}{ccc} & k & \\ u_A \swarrow & & \searrow u_B \\ A & \xrightarrow{f} & B \end{array}$$

$$f(a-b) = f(a) - f(b)$$

$$f(u_A(u)) = u_B(u)$$

② $g: C \rightarrow D$ is a coalg hom:

$$\begin{array}{ccc} D \otimes D & \xleftarrow{g \otimes g} & C \otimes C \\ \Delta_D \uparrow & & \uparrow \Delta_C \\ D & \xleftarrow{g} & C \end{array}$$

and

$$\begin{array}{ccc} & k & \\ \epsilon_D \swarrow & & \swarrow \epsilon_C \\ D & \xleftarrow{g} & C \end{array}$$

③ $I \triangleleft C: \Delta(I) \subseteq I \otimes C + C \otimes I$ and $\epsilon(I) = 0$

$\Leftrightarrow \exists f$ be a coalg. hom s.t. $I = \text{Ker} f$

1st. iso. thm: $C/\text{Ker} f \cong \text{Im} f$

Remark: In general: $M' \subseteq M, N' \subseteq N \not\Rightarrow M' \otimes_R N' \hookrightarrow M \otimes N$

$$\text{Example: } \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$$

However, \mathbb{K} is a field, $\Rightarrow U' \otimes_{\mathbb{K}} V' \hookrightarrow U \otimes V$ (pf omitted)

3. others: twist (flip) τ

opposite: op, cop

commutative, cocommutative

Note: coideal 从作用上, 类似 ideal, 但对偶性质上类似 subalg.

Ch 1.2. Dual of coalg and alg.

1. transpose

$$\langle \cdot, \cdot \rangle: V^* \otimes V \rightarrow \mathbb{K}, \quad (V^* = \text{Hom}(V, \mathbb{K}))$$

$$v^* \otimes v \mapsto v^*(v)$$

$$\text{i.e. } \langle v^*, v \rangle := v^*(v)$$

$$\textcircled{2} f: V \rightarrow W \in \text{Hom}(V, W)$$

$$\Rightarrow {}^t f: W^* \rightarrow V^* \in \text{Hom}(W^*, V^*)$$

$f^*: W^* \mapsto W^* \circ f$

i.e. $t: \text{Hom}(V, W) \rightarrow \text{Hom}(W^*, V^*) \rightarrow \text{Hom}(V^{**}, W^{**})$

$$f \mapsto f^* \begin{pmatrix} W^* \rightarrow V^* \\ W^* \mapsto W^* \circ f \end{pmatrix}$$

$$\textcircled{3} \langle {}^t f \cdot w^*, v \rangle = \langle w^*, f(v) \rangle$$

Note: t 为嵌入, 使用 \langle, \rangle 在处理对偶时, 记号不混乱.

2. Lemma: $U \xrightarrow{f} V$ is commutative

$$\Leftrightarrow U^* \xleftarrow{f^*} V^*$$

is commutative

$$\text{pf: } \Rightarrow \langle f^* \cdot g^* w^*, u \rangle = \langle g^* w^*, f(u) \rangle, \forall u \in U$$

$$= \langle w^*, g(f(u)) \rangle$$

$$= \langle (g \circ f)^* w^*, u \rangle$$

$$\therefore f^* \cdot g^* = (g \circ f)^*$$

\Leftarrow : t is injective.

3. $\textcircled{1}$ only $\xrightarrow{\text{dual}}$ only

$$\begin{array}{ccc} C \otimes C \otimes C & \xleftarrow{I_C \otimes \Delta} & C \otimes C \\ \Delta \otimes I_C \uparrow & & \uparrow \Delta \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

\Leftrightarrow

$$\begin{array}{ccc} (C \otimes C \otimes C)^* & \xrightarrow{(I_C \otimes \Delta)^*} & (C \otimes C)^* \\ (\Delta \otimes I_C)^* \downarrow & & \downarrow \Delta^* \\ (C \otimes C)^* & \xrightarrow{\Delta^*} & C^* \end{array}$$

$$\begin{array}{ccc}
 C^* \otimes C^* & \xrightarrow{I_C \otimes \Delta^*} & C^* \otimes C^* \\
 \Delta^* \otimes 1_C \downarrow & & \downarrow \Delta^* \\
 C^* \otimes C^* & \xrightarrow{\Delta^*} & C^*
 \end{array}
 \quad \S
 \quad
 \begin{array}{ccc}
 & C^* \otimes C^* & \\
 \epsilon \otimes 1_C \nearrow & \downarrow \Delta^* & \nwarrow 1_C \otimes \epsilon^* \\
 1_C \otimes C^* & C^* & C^* \otimes 1_C
 \end{array}$$

where: $C^* \otimes C^* \hookrightarrow (C \otimes C)^*$

$$u^* \otimes v^* \mapsto \begin{cases} C \otimes C \rightarrow K \\ n \otimes v \mapsto u^*(u \cdot v^*(v)) \end{cases}$$

then $(C^*, m, n) = (C^*, \Delta^*/C^* \otimes C^*, \epsilon^*)$ is an alg.

② $m: C^* \otimes C^* \rightarrow C^*$

$$c^* \otimes d^* \mapsto m(c^* \otimes d^*)$$

$$\begin{aligned}
 \langle m(c^* \otimes d^*), c \rangle &= \langle \Delta^*(c^* \otimes d^*), c \rangle \\
 &= \langle c^* \otimes d^*, \Delta(c) \rangle
 \end{aligned}$$

Note: $K, G \not\subseteq, (c^* \otimes d^*) \circ \Delta(g) = (c^* \otimes d^*)(g \otimes g), \forall g \in G$

$$\begin{aligned}
 &= c^*(g) \cdot d^*(g) \\
 \Rightarrow (c^* \otimes d^*) \circ \Delta(2g) &= (c^* \otimes d^*)(2g \otimes 2g) \\
 &= 2 \cdot c^*(g) \cdot d^*(g) \\
 &\neq (c^* \otimes d^*)(2g \otimes 2g)
 \end{aligned}$$

③ alg \rightarrow walg.

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{1_A \otimes m} & A \otimes A & & (A \otimes A \otimes A)^* & \xleftarrow{(1_A \otimes m)^*} & (A \otimes A)^* \\
 m \otimes 1_A \downarrow & \hookrightarrow & \downarrow m & \Leftrightarrow & (m \otimes 1_A)^* \uparrow & \hookrightarrow & \uparrow m^* \\
 A \otimes A & \xrightarrow{m} & A & & (A \otimes A)^* & \xleftarrow{m^*} & A^*
 \end{array}$$

Remark: " $m^*(A^*)$ 未必在 $A^* \otimes A^*$ 中, 令 $A^0 = m^{*-1}(A^* \otimes A^*) \subseteq A^*$

使 A^0 作成子代数且在 A^* 中极大.

" $A^0 = \{ f \in A^* \mid \exists I \text{ be a wfinite ideal of } A \text{ s.t. } f(I) = 0 \}$

Cpt 1.3. Bialgebras

1. A vec. space B is a bialgebra.

if $\begin{cases} (B, \Delta, \epsilon) \text{ is a coalg.} \\ (B, m, u) \text{ is an alg.} \end{cases}$ and Δ and ϵ are alg. homs.
 or m and u are coalg. homs

2. A, B 为代数 $\Rightarrow A \otimes B, A \otimes B$ 为代数

具体地, $m: (A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B$ $u: K \rightarrow A \otimes B$

$$(a_1, b_1) \otimes (a_2, b_2) \mapsto (a_1 a_2, b_1 b_2) \quad 1 \mapsto (1, 1)$$

或 $m: (A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B$ $u: K \rightarrow A \otimes B$

$$(a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto (a_1 a_2) \otimes (b_1 b_2) \quad 1 \mapsto 1 \otimes 1$$

C, D 为代数 $\Rightarrow C \otimes D, C \otimes D$ 为代数.

具体地, $\Delta: C \otimes D \rightarrow (C \otimes D) \otimes (C \otimes D)$, $\epsilon: C \otimes D \rightarrow K$

$$c \otimes d \mapsto (c \otimes 1) \otimes (1 \otimes d) \quad c \otimes d \mapsto \epsilon(c) \cdot \epsilon(d)$$

或 $\Delta: C \otimes D \rightarrow (C \otimes D) \otimes (C \otimes D)$, $\epsilon: C \otimes D \rightarrow K$

$$(c, d) \mapsto (c \otimes 1) \otimes (1 \otimes d) \quad (c, d) \mapsto \epsilon(c) + \epsilon(d)$$

3. Δ, ϵ are alg. homs $\Leftrightarrow m, u$ are coalg. homs

pf: \Rightarrow

$$0: B \rightarrow B \otimes B \Leftrightarrow m \downarrow \quad B \otimes B \xrightarrow{\Delta \otimes \Delta} (B \otimes B) \otimes (B \otimes B) \downarrow m_{B \otimes B}$$

is an alg. hom

$$B \xrightarrow{\Delta} B \otimes B$$

$$\text{and } \begin{array}{ccc} & K & \\ u \swarrow & & \searrow u \otimes u \\ B & \xrightarrow{\Delta} & B \otimes B \end{array}$$

$\epsilon: B \rightarrow K$
is an alg. hom

$$\Leftrightarrow \begin{array}{ccc} B \otimes B & \xrightarrow{\epsilon \otimes \epsilon} & K \otimes K \\ m \downarrow & & \downarrow m_{K \otimes K} \\ B & \xrightarrow{\epsilon} & K \end{array} \quad \text{and } \begin{array}{ccc} & K & \\ u \swarrow & & \searrow u_K \\ B & \xrightarrow{\epsilon} & K \end{array}$$

$$\Leftrightarrow \begin{array}{ccc} B \otimes B & \xrightarrow{\epsilon \otimes \epsilon} & K \\ m \downarrow & & \downarrow \epsilon \\ B & \xrightarrow{\epsilon} & K \end{array} \quad \text{and } \epsilon \circ u = 1_K$$

12)

$$m : B \otimes B \rightarrow B \quad \Leftrightarrow \quad \begin{array}{ccc} B \otimes B & \xrightarrow{\Delta_{B \otimes B}} & (B \otimes B) \otimes (B \otimes B) \\ \downarrow m & & \downarrow m \otimes m \\ B & \xrightarrow{\Delta} & B \otimes B \end{array}$$

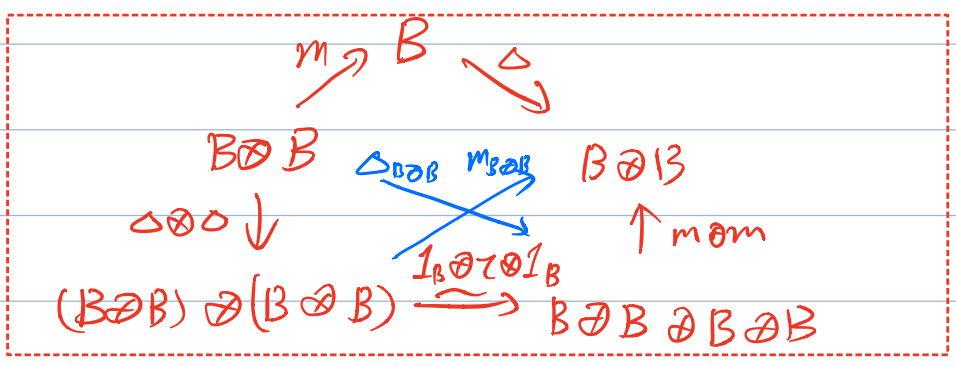
is a weak hom

and.

$$\begin{array}{ccc} & K & \\ \epsilon \otimes \epsilon \nearrow & & \nwarrow \epsilon \\ B \otimes B & \xrightarrow{m} & B \end{array}$$

Similarly, $u : K \rightarrow B$ is a weak hom $\Leftrightarrow \begin{array}{ccc} & K & \\ u \swarrow & & \searrow u \otimes u \\ B & \xrightarrow{\Delta} & B \otimes B \end{array}$ and $\epsilon \circ u = 1_K$

Noticed that:
the first diagram
of m and Δ
is equivalent to



Remark: Equivalent condition (straight from the proof)

- (1) $\epsilon \circ u = 1_K$, i.e. $\epsilon(1) = 1$ (unit)
- (2) $\epsilon \circ m = \epsilon \otimes \epsilon$, i.e. $\epsilon(a \cdot b) = \epsilon(a) \cdot \epsilon(b)$
- (3) $\Delta \circ u = u \otimes u$, i.e. $\Delta(u) = | \otimes |$
- (4) $\Delta \circ m = m \otimes m \circ \Delta_{B \otimes B}$, i.e. $(c_{11} d_{11}) \otimes (c_{12} d_{12}) = (cd)_{11} \otimes (cd)_{12}$

4. 1st. hom. thm.

- ⊙ bialgebra map: both an alg. and coalg. morphism.
- bi-ideal: both an alg. and coalg. ideal.

⊕ $I \triangleleft B \Leftrightarrow \exists f : B \rightarrow B'$ bialg. map. s.t. $I = \text{Ker } f$
 $B / \text{Ker } f \cong \text{Im } f$.

5. Examples:

① grp alg. KG . $(\Delta(g)) = g \otimes g$

②. let \mathfrak{g} be a K -Lie alg. $B = U(\mathfrak{g})$ be its universal Enveloping algebra. let $\Delta(x) = 1 \otimes x + x \otimes 1$, $\epsilon(x) = 0$, both KG and $U(\mathfrak{g})$ are cocommutative.

6. ①. grouplike element: $g \in C$ s.t. $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$
 $G(C) = \{g \in C \mid g \text{ is a grouplike element}\}$ is a subalg.

②. for $g, h \in G(C)$, $c \in C$ is called g, h -primitive element if $\Delta(c) = c \otimes g + h \otimes c$

$P_{g,h}(C) = \{c \in C \mid c \text{ is } g, h\text{-primitive}\}$

Note: $c \in P_{g,h}(C) \Rightarrow \epsilon(cc) = \epsilon \otimes \epsilon \circ \Delta(cc) = \epsilon(cc) \cdot \epsilon(g) + \epsilon(h) \cdot \epsilon(cc) = 2\epsilon(cc)$
 $\Rightarrow \epsilon(cc) = 0$.

$P_{g,h}(C) \triangleleft C$, since $\Delta(P_{g,h}(C)) \subseteq P_{g,h}(C) \otimes C + C \otimes P_{g,h}(C)$.

In fact, any subspace of $P_{g,h}(C)$ is an ideal of C .

③ B is a bialgebra, $P(B) := P_{1,1}(B)$

Elements of $P(B)$ are called primitive.

7. $B = (KG \Rightarrow G(B) = G$

$B = U(\mathfrak{g})$, $\text{char } K = 0 \Rightarrow P(B) = \mathfrak{g}$

$B = U(\mathfrak{g})$, $\text{char } K = p \Rightarrow P(B) = \text{span} \{x^{p^k} \mid k \geq 0, x \in \mathfrak{g}\}$

which is a restricted p -lie alg.

let A be an algebra

define $\text{Alg}(A, K) = \{f \in A^* \mid f \text{ is an alg. map}\}$

then. $\text{Alg}(A, K) = G(A^0)$ (Cpt 4)

8. B is an bialg $\Rightarrow B^o$ is an bialg. (Cpt 4)

B is bialgebra $\Rightarrow B^o, B^{op}, B^{op, cop}$ are bialgebras

Cpt. 4. Convolution and Sweedler notation.

By default, C is a coalgebra and A is an algebra.

1. convolution (\mathbb{K} field)

① $(\text{Hom}_{\mathbb{K}}(C, A), m, \eta)$ is an algebra.

where: $m = * : \text{Hom}_{\mathbb{K}}(C, A) \otimes \text{Hom}_{\mathbb{K}}(C, A) \rightarrow \text{Hom}_{\mathbb{K}}(C, A)$

$$f \otimes g \mapsto m \circ (f \otimes g) \circ \Delta$$

$*$ is called the convolution product

$\eta \circ \epsilon$ is its unit.

② $C^* = \text{Hom}(C, \mathbb{K})$ is an algebra.

$$f, g \in C^* \Rightarrow f * g(C) = m \circ (f \otimes g) \circ \Delta(C)$$

③ anti-convolution

$$x : \text{Hom}_{\mathbb{K}}(C, A) \otimes \text{Hom}_{\mathbb{K}}(C, A) \rightarrow \text{Hom}_{\mathbb{K}}(C, A)$$

$$f \otimes g \mapsto m \circ (f \otimes g) \circ \tau \circ \Delta$$

2. Sweedler notation.

$$\textcircled{1} \Delta : C \rightarrow C \otimes C \Rightarrow \Delta(c) = \sum_{i=1}^r u_i \otimes v_i$$

write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, for short and \sum can be omitted.

② coassociativity

$$(\mathcal{I}_C \otimes \Delta) \circ \Delta(c) = (\Delta \otimes \mathcal{I}_C) \circ \Delta(c)$$

$$\Leftrightarrow (\mathcal{I}_C \otimes \Delta)(c_{(1)} \otimes c_{(2)}) = (\Delta \otimes \mathcal{I}_C)(c_{(1)} \otimes c_{(2)})$$

$$\Leftrightarrow c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$$

$$\text{write } \Delta_2 = (\mathcal{I}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathcal{I}_C) \circ \Delta$$

then $\Delta_2(c) = c_{(1)} \otimes c_{(1)} \otimes c_{(2)}$ is well defined by coasso.

$$\textcircled{3} \Delta^i = \mathcal{I}_C \otimes \overset{(i)}{\dots \otimes \Delta \otimes \dots} \otimes \mathcal{I}_C \text{ (} i \text{ items in all)}$$

$\Delta_n = \Delta^n \otimes \dots \otimes \Delta'$ is independent of the choice of $\{j\}$.

write $\Delta_n(c) = c_{(1)} \otimes \dots \otimes c_{(n)}$ for short.

3. Example: $C_{(1)} \otimes C_{(2)} \otimes C_{(2)} \otimes C_{(1)} \otimes C_{(3)} \otimes C_{(2)} \otimes C_{(3)}$

4. property: define $\epsilon^i = I_C \otimes \dots \otimes \epsilon \otimes \dots \otimes I_C$ (i items in all)
 then $C^{nH} \circ \Delta_n = \Delta_{n-1}$

pf: unit: $(\epsilon \otimes I_C) \circ \Delta = I_C = (I_C \otimes \epsilon) \circ \Delta$

$\Leftrightarrow \epsilon(C_{(1)}) \cdot C_{(2)} = C_{(1)} \cdot \epsilon(C_{(2)})$

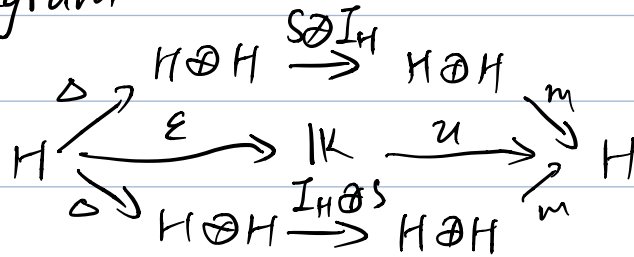
5. $(f * g)(c) = m \circ (f \otimes g) \circ \Delta(c) = f(C_{(1)}) \cdot g(C_{(2)})$

Remark: S/λ Sweedler notation 便于推导 & 体现式子对称性

Opt 1.5. Antipode and Hopf alg.

1. ① Hopf alg. $\left\{ \begin{array}{l} (H, m, u, \Delta, \epsilon) \text{ is an bialgebra} \\ \exists S \in \text{Hom}_K(H, H) \text{ s.t. } S * I_H = I_H * S = u \circ \epsilon \end{array} \right.$

② diagram



③ Sweedler: $S(h_{(1)})h_{(2)} = \epsilon(h) \cdot I_H = h_{(1)} \cdot S(h_{(2)})$

I_H 与 I_H 前边弄错了!

2. $f: H \rightarrow K$ is called a Hopf morphism

if it is a bialg. hom. and $f(S_H h) = S_K f(h)$

i.e.
$$\begin{array}{ccc} H & \xrightarrow{f} & K \\ S_H \downarrow & f & \downarrow S_K \\ H & \xrightarrow{f} & K \end{array}$$

I is a Hopf ideal if I is a bideal and $S I \subseteq I$

3. Examples.

① $H = K.G$:
$$\begin{array}{ccc} g \otimes g & \xrightarrow{\epsilon} & S(g) \cdot g \\ \searrow \epsilon & \downarrow u & \downarrow I_H \\ g \otimes g & \xrightarrow{\epsilon} & g \cdot S(g) \end{array}$$
 Corollary: $g \in G(H) \Rightarrow S_H(g) = g^{-1}$

②. $H = U(g)$

$$\begin{array}{c}
 g \otimes 1 + 1 \otimes g \rightarrow S(g) + g \\
 \begin{array}{ccc}
 \nearrow & \xrightarrow{\epsilon} & 0 \xrightarrow{u} 0 \\
 \searrow & & \\
 g \otimes 1 + 1 \otimes g & \rightarrow & g + S(g)
 \end{array}
 \end{array}$$

Corollary: $g \in P(H) \Rightarrow S_H(g) = -g$

③ H is a Hopf alg $\Rightarrow H^o$ is also a Hopf alg. (Cpt 9)
with antipode S^*

④. $H_q = \mathbb{K} \langle 1, g, x, gx \rangle$

$$\begin{array}{l}
 \text{mul: } g^2 = 1, x^2 = 0, xg = -gx \\
 \text{comul: } \Delta g = g \otimes g, \Delta x = x \otimes 1 + g \otimes x \\
 \epsilon(g) = 1, \epsilon(x) = 0
 \end{array}$$

Ex: determine S_H and find its order

Remark: $\dim(S_H) = 2n$ or infinite.

⑤. $B = O(M_n(\mathbb{K})) = \mathbb{K} [X_{ij} \mid 1 \leq i, j \leq n] = \mathbb{K} \langle X_{ij}^{kr} \mid k, r \in \mathbb{Z}_n \rangle$

mul: polynomial ring

comul: $\Delta(X_{ij}) = \sum_{k=1}^n X_{ik} \otimes X_{kj}, \epsilon(X_{ij}) = \delta_{ij}$

let $X = (X_{ij})_{n \times n}$, then $\det X \in G(B)$ ($n=1, 2, \dots$)

$\det X = \sum_{\sigma \in S_n} (-1)^{\text{sgn } \sigma} \cdot X_{1\sigma(1)} \cdots X_{n\sigma(n)}$ is irreversible

contradictory to " $g \in G(H) \Rightarrow S_H(g) = g^{-1}$ ".

closely hopf algs related to B :

$U(M_n(\mathbb{K})) / (\det X - 1)$

$O(M_n(\mathbb{K})) [\det X^{-1}]$

4. Prop: S is both anti-alg. and anti-coalg. (Sweedler Cpt 6)

i.e. $S \circ m = m \circ (S \otimes S) \circ \tau$

$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$

or: $S(h \cdot k) = S(k) \cdot S(h), S(1) = 1$

$S(h_{(2)}) \otimes S(h_{(1)}) = S(h)_{(1)} \otimes S(h)_{(2)}$

$$5. (H, S) \Rightarrow (H^{op, cop}, \bar{S})$$

where \bar{S} is the inverse of 1_H under twisted convolution.

$$(f * g)(c) = f(c_{(1)}) \cdot g(c_{(2)}), (f \times g)(c) = f(c_{(1)}) \cdot g(c_{(2)})$$

$$\bar{S}(h_{(2)}) \cdot h_{(1)} = h_{(2)} \cdot \bar{S}(h_{(1)}) = \epsilon(h) \cdot 1_H$$

6. B is a bra algebra, then B is a Hopf algebra with (composition)

invertible antipode $S \Leftrightarrow B^{cop}$ is $\dots \dots \bar{S}$

in this case, $\bar{S} = S^{-1}$ (i.e. both \circ and $*$)

pf: omitted. 笑死我了!

Corollary: H is cocommutative or commutative $\Rightarrow S^2 = Id$.

Remark: $S^2 = id$ 相关. 10 是已. $G \in 2$ 中 讨论

在 \mathbb{F} , Hopf 基本性质. $\left\{ \begin{array}{l} \text{alg} \\ \text{coalg} \end{array} \right. + \begin{array}{l} m, \mu \\ \text{or} \\ \Delta, \epsilon \end{array} + \text{antipode}$

pf: Example: $A^0 = A^$*

$$A = \mathbb{K} \{ a_i \mid i \in \mathbb{I} \} \text{ with } 1 = a_0 \in \mathbb{I}$$

$$m: A \otimes A \rightarrow A$$

$$\epsilon: \mathbb{K} \rightarrow A$$

$$a_i \otimes a_j \mapsto \begin{cases} 1, & i=j=0 \\ 0, & \text{other cases} \end{cases}$$

$$1 \mapsto a_0$$

let $I = \text{span} \{ a_i \mid i \neq 0 \}$ be an cofinite ideal of A

then any subspace of I is also an ideal of A

$\forall f \in A^*$, $\text{Ker} f$ is cofinite $\Rightarrow I \cap \text{Ker} f$ is a cofinite ideal of A

$$\Rightarrow f \in A^0 \quad \#$$

In fact, $\Delta = m^*: A^* \rightarrow (A \otimes A)^*$

$$a^* \mapsto \langle \Delta(a^*), - \rangle$$

we prove that $\text{Im } \Delta \subseteq A^* \otimes A^*$,

$$\text{i.e. } \langle \Delta(a^*), - \rangle = \langle a_{ij}^* \otimes a_{kl}^*, - \rangle$$

$$\langle \Delta(a^*), u \otimes v \rangle = \langle m^*(a^*), u \otimes v \rangle, \quad u, v \in \text{Basis of } A.$$

$$= \langle a^*, m(u \otimes v) \rangle$$

$$= \langle a^*, u \cdot v \rangle$$

$$= \begin{cases} a^*(1), & u=v=1 \\ a^*(u), & u \neq 1, v=1 \\ a^*(v), & u=1, v \neq 1 \\ 0, & u \neq 1 \neq v \end{cases}$$

$$= \begin{pmatrix} a^*(u) \cdot 1^*(v) + 1^*(u) \cdot a^*(v) \\ -a^*(u) \cdot 1^*(u) \cdot 1^*(v) \end{pmatrix}$$

$$= \langle a^* \otimes 1^* + 1^* \otimes a^* - a^*(1) \cdot 1^* \otimes 1^*, u \otimes v \rangle$$

hence $\Delta: A^* \rightarrow A^* \otimes A^*$

$$a^* \mapsto a^* \otimes 1^* + 1^* \otimes a^* - a^*(1) \cdot 1^* \otimes 1^*$$

Example. $A^0 = 0$.

let $A = k\langle t \rangle$ be an infinite-dimensional alg.

then $I \triangleleft A \Rightarrow I = 0$ or A

$$\Rightarrow A^0 = 0$$