1.6 - 1.9. representation of hopf alg.

Cpt 1.6. Modules and comodules

By default: A is an algebra over $\mathbb{k}$.
C is a coalgebra over $\mathbb{k}$.

1. defs

① recall that, M is an (left) A-module

if $\begin{cases} (a\cdot b)m = a\cdot(b\cdot m) \\ 1_A\cdot m = m \end{cases}$ and $\left.\begin{cases} (a+b)m = am+bm \\ a(m+n) = am+an \\ (k\cdot a)\cdot m = k\cdot(am) = a\cdot km \end{cases}\right\}$ bilinearity. (omitted)

Equivently, let $\gamma: A\otimes M \to M$ be a $\mathbb{k}$-linear map

we say that (M,γ) is a left A-module

if the following diagrams commute (TFDC for short)

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "m\otimes I_M"] \arrow[d, "I_M\otimes"'] & A\otimes M \arrow[d, "\gamma"] \\ A\otimes M \arrow[r, "\gamma"] & M \end{tikzcd} , \quad \begin{tikzcd} \mathbb{k}\otimes M \arrow[r, "u\otimes I_M"] \arrow[dr, equal, "\backsimeq"'] & A\otimes M \arrow[d, "\gamma"] \\ & M \end{tikzcd}$$

The cat. of left A-module is denoted by ${}_A\mathcal{M}$

②. let $\rho: M \to M\otimes C$ be a $\mathbb{k}$-linear map

we say that (M,ρ) is a (right) C-comodule

if TFDC:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M\otimes C \arrow[d, "I_M\otimes\Delta"] \\ M\otimes C \arrow[r, "\rho\otimes I_M"] & M\otimes C\otimes C \end{tikzcd} , \quad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, equal, "\backsimeq"'] & M\otimes C \arrow[d, "I_M\otimes\varepsilon"] \\ & M\otimes\mathbb{k} \end{tikzcd}$$

coassociaty $\qquad\qquad$ counit

The cat. of right C-module is denoted $\mathcal{M}^C$

③ Sweedler notations.

let (M, ρ), be a right C-module

we write $\rho(m) = m_0 \otimes m_1 \in M \otimes C$

Analogously, for a left C-module (M, ρ')

we write $\rho'(m) = m_{-1} \otimes m_0 \in C \otimes M$

Note: coasso: $m \to m_0 \otimes m_1 \to m_0 \otimes (m_1)_1 \otimes (m_1)_2$

$\searrow$ $m_0 \otimes m_1 \to (m_0)_1 \otimes (m_0)_2 \otimes m_1$

counit: $m = m_0 \cdot \epsilon(m_1)$

That is to say, sweedler notation on C and M are compatible.

$\rho^2 = (\rho \otimes I_C) \circ \rho$ $\Rightarrow \rho^n(m) = m_0 \otimes m_1 \cdots \otimes m_n$

$= (I_M \otimes \Delta) \circ \rho$

$(I_M \otimes \epsilon) \circ \rho = I_M \Rightarrow \epsilon$ can reduce the multiplicity of ρ^n.

Remark: (A, B)-bimodule (M, γ_A, γ_B)

= left A + right B + associaty

here asso. means $(am)b = a(mb)$.

or

$$\begin{tikzcd} A \otimes M \otimes B \arrow[r] \arrow[d] & M \otimes B \arrow[d] \\ A \otimes M \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & M \end{tikzcd}$$

One can define (C, D)-cobimodule (M, ρ_C, ρ_D) in a similar way

In particular, A is a (A, A)-bimodule in itself

C is a (C, C)-cobimodule in itself.

2. lemma.

① M is a right C-comodule $\Rightarrow M$ is a left C^*-module.

pf: let $\gamma: C^* \otimes M \to M$

$c^* \otimes m \mapsto c^* \rightharpoonup m$ (Radford) where $c^* \rightharpoonup m = c^*(m_1) \cdot m_0$

unit : $\epsilon \rightharpoonup m = \epsilon(m_1) \cdot m_0$

asso : $a^* \rightharpoonup (b^* \rightharpoonup m) = a^* \rightharpoonup (b^*(m_1) \cdot m_0)$

$= b^*(m_2) \cdot a^*(m_1) \cdot m_0$

$(a^* \cdot b^*) \rightharpoonup m = (a^* \cdot b^*)(m_1) \cdot m_0$ sweedler.

$= a^*(m_1) \cdot b^*(m_2) \cdot m_0$

② A left A-module is called locally finite if $\dim A \cdot m < \infty$, $\forall m \in M$.

Prop: If M is a left A-module, then

M is a right A°-comodule $\Leftrightarrow$ M is a locally finite A-mod.

recall: $A^\circ = \{ f \in A^* \mid \operatorname{Ker} f$ contain a cofinite ideal of $A \}$

pf: $\Leftarrow$: let $\{m_1, \dots, m_n\}$ be a basis of $A \cdot m$

$\forall a \in A$. $a \cdot m = \sum_i f_i(a) \cdot m_i$ for some $f_i(a) \in \mathbb{K}$

let $I = \operatorname{Ker}(A \to \operatorname{End}_{\mathbb{K}} M)$

$= \{ a \in A \mid a \cdot m_i = 0, \forall i \}$

I is a cofinite ideal of A since $\dim \operatorname{End}_{\mathbb{K}} M < \infty$

$f_i(I) = 0 \Rightarrow I \in A^\circ$

$\Rightarrow (M, \rho)$ becomes an right A°-comodule

via $\rho : M \to M \otimes A^\circ$

$m \mapsto \sum_i m_i \otimes f_i$

$\Rightarrow$: M is a right A°-comodule

$\Rightarrow$ $a \cdot m = m_{(0)} \cdot m_{(1)}(a) \in \operatorname{span} m_{(0)}$

Remark: When saying that M has A-module and A°-comodule structures

we require that $a \cdot m = m_{(0)} \cdot m_{(1)}(a)$ $\in A^\circ$ (or $c^* \cdot m = m_{(0)} \cdot c^*(m_{(1)})$ $\in C$)

③. left C^*-module $\not\Rightarrow$ right C-comodule

A C^*-module M which becomes a C-comodule in the natural way is called rational.

Note: $\{$ right C-comodule $\} \xrightarrow{\sim} \{$ rational C^*-module $\}$

3. Examples

① If (C, Δ) is a right C-comodule.

then C is a left C^* module.

more precisely, $f \rightharpoonup c = \langle f, c_1 \rangle \cdot c_0 = \langle f, c_2 \rangle \cdot c_1$

Since C and C^* are both C^*-module.

we write (Radford) $l: C^* \to \mathrm{End}\, C^*$ (left multiplication)

$r: C^* \to \mathrm{End}\, C^*$ (right multiplication)

$R: C^* \to \mathrm{End}\, C$. ($C$ is a right-C^*-mod)

$L: C^* \to \mathrm{End}\, C$, ($C$ is a left-C^*-mod)

in fact, $\langle g, f \rightharpoonup c \rangle = \langle f, c_2 \rangle \cdot \langle g, c_1 \rangle$

$= (g * f)(c)$

$= \langle gf, c \rangle$

i.e. $\langle g, L(f)(c) \rangle = \langle r(f)(g), c \rangle$

therefore $r(f) \in \mathrm{End}\, C^*$ is the transpose of $L(f)$

In general, TFDC:

$$\begin{tikzcd} C^* \arrow[r, "L"] \arrow[dr, "r"'] & \mathrm{End}\, C \arrow[d, "t"] \\ & \mathrm{End}\, C^* \end{tikzcd} , \quad \begin{tikzcd} C^* \arrow[r, "R"] \arrow[dr, "l"'] & \mathrm{End}\, C \arrow[d, "t"] \\ & \mathrm{End}\, C^* \end{tikzcd}$$

② $I \subseteq A$ is a left ideal of A

$\overset{\text{def}}{\iff}$ I is a left A-mod

$\overset{\text{def}}{\iff}$ $A \cdot I \subseteq I$, i.e. $m(A \otimes I) \subseteq I$

Analogously,

I is a right coideal of C

$\overset{def}{\Longleftrightarrow}$ I is a right comodule of C

$\overset{def}{\Longleftrightarrow}$ $m(I) \subseteq I \otimes C$

Remark: most of the defs and props have similar argument on left cases and right cases

we focus on one case in the following text.

③. every left-C^* submodule of C is a right C-subcomodule

i.e. C^*-submodules of C are rational.

Pf: provided V is a left C^*-mod

$\forall c \in V$, let $\Delta(c) \overset{r}{=} \sum_{i=1}^{r} u_i \otimes v_i \in C \otimes C$ (r for reduced)

let $u^i \in C^*$ s.t. $u^i(u_j) = \delta_{ij}$

then $v_i = c \leftharpoonup u^i \in V$

hence $\mathrm{Span}\{v_i\} \subseteq V$. i.e. $\Delta(c) \in C \otimes V$ #

④ "A is a left A-mod. diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathrm{End}\, A \arrow[d, "\in"] \\ & \mathrm{End}\, A^* \end{tikzcd}$$

make A^* into a right A-mod

more precisely, $\langle f \leftharpoonup a, b \rangle = \langle f, a \cdot b \rangle$

(2) A^0 is a sub A-module of A^*

Pf: $f \in A^0 \Rightarrow \exists I \lhd A$ s.t. $\mathrm{codim}\, I < \infty$, $f(I) = 0$

$\Rightarrow \langle f \leftharpoonup a, b \rangle = \langle f, a \cdot I \rangle \subseteq \langle f, I \rangle = 0$

$\Rightarrow f \leftharpoonup a \in A^0$

⑤. let $C = \mathbb{K}G$. M is a right C-comod iff M is a G-graded module i.e. $M = \bigoplus_{g \in G} M_g$.

Pf: $\Rightarrow$:

$m \xrightarrow{\rho} \sum_i m_i \otimes g_i \xrightarrow{I_M \otimes \Delta} \sum_i m_i \otimes g_i \otimes g_i$

$\rho \Rightarrow \sum_i m_i\otimes g_i \xrightarrow{\rho\otimes I_C} \sum_{i,j}(m_{ij}\otimes g_j)\otimes g_i$

$\sum_i (m_i\otimes g_i - \sum_j m_{ij}\otimes g_j)\otimes g_i = 0$

$\Rightarrow (m_i\otimes g_i - \sum_j m_{ij}\otimes g_j) = 0$

$\Rightarrow \sum_j (m_{ij} - \delta_{ij} m_i)\otimes g_j = 0$

$\Rightarrow m_{ij} = \delta_{ij} m_i$,

$\Rightarrow \rho(m_i) = m_i\otimes g_i$

let $M_g = \{m\in M \mid \rho(m) = m\otimes g\}$

Since $m \in \operatorname{span}\{m_i\} \subseteq \sum_i M_{g_i}$ we have $M = \sum_g M_g = \bigoplus_g M_g$

$\Leftarrow$: let $\rho: M \to M\otimes C$, $m_g \mapsto m_g\otimes g$, it's trivial to check (M,ρ) is a C-comod.

⑥. let $A = \mathbb{K}G$, M be an A-mod, by the lemma above M is a right A°-comodule iff M is a locally finite A-mod.

i.e. $A.m = G.m$ is finie-dimensional.

or: the G-action on M is locally finite.

Cpt1.7. inv. and coinv.

1. defs

① let M be a left H-mod. the (left) invariants of H on M are the set

$$M^H = \{m\in M \mid h.m = \varepsilon(h).m,\ \forall h\in H\}$$

i.e. M^H are elements of M, s.t. TFDC

$$\begin{tikzcd} H\otimes M \arrow[r, "\gamma"] \arrow[d, "\varepsilon\otimes \mathrm{Id}"'] & M \\ \mathbb{K}\otimes M \arrow[ur, phantom, "//"] & \end{tikzcd}$$

② let M be a right H-comod. the coinvariants of H

on M are the set

$$M^{coH} = \{ m \in M \mid \rho(m) = m \otimes 1 \}$$

i.e. M^{coH} are elements of M s.t. TFDC

$$\begin{tikzcd} M \otimes H & M \arrow[l, "\rho"'] \arrow[dl, equal] \\ M \otimes \mathbb{K} \arrow[u, "I_M \otimes u"] & \end{tikzcd}$$

2. Lemma.

① let M be a right H-comod, consider its left H^*-mod structure we have $M^{H^*} = M^{coH}$

② Analogously, let M be a left H-mod, s.t. it is also a right H^0-comod, then $M^H = M^{coH^0}$

pf: ①. $m \in M^{H^*} \Leftrightarrow h^* \rightharpoonup m = \epsilon(h^*) \cdot m \quad , \forall h^* \in H^*$

$\Leftrightarrow h^*(m_1) \cdot m_0 = \epsilon(h^*) \cdot m \quad , \forall h^* \in H^*$

$\Leftrightarrow m_0 = m, \quad h^*(m_1) = \langle u^*(h^*), 1_{\mathbb{K}} \rangle, \forall h^* \in H^*$

$\Leftrightarrow m_0 = m, \quad m_1 = 1$

$\Leftrightarrow \rho(m) = m \otimes 1$

where $h^*(m_1) = \langle u^*(h^*), 1_{\mathbb{K}} \rangle = \langle h^*, u(1) \rangle = \langle h^*, 1_H \rangle$,

i.e. $h^*(m_1) = h^*(1_H) \quad \forall h^* \in H$

hence $m_1 = 1_H$

②. the same as ①.

Note: "(1)" $u: \mathbb{K} \to A \Rightarrow A^* \xrightarrow{u^*} \mathbb{K}^*$, $\pi: \mathbb{K}^* \to \mathbb{K}$

$$\begin{tikzcd} A^* \arrow[r, "u^*"] \arrow[dr, "\epsilon"'] & \mathbb{K}^* \arrow[d, "\pi", "\wr"'] \\ & \mathbb{K} \end{tikzcd} \qquad 1^* \mapsto 1$$

$\epsilon(h^*) = \pi \circ u^*(h^*) = \langle u^*(h^*), 1 \rangle$

"(2)" right C-comod $\Rightarrow$ left C^*-mod, $a^* \rightharpoonup m = a^*(m_1) \cdot m_0$

3. Examples

① let M be any vector space (trivial)

$\rho: M \to M\otimes H$ makes M into a right H-comod.

$m \mapsto m\otimes 1$

(trivial)

Similarly, $\gamma: H\otimes M \to M$ make M a left H-mod.

$h\otimes m \mapsto \epsilon(h)\cdot m$

② let $H = \Bbbk G$, if M is a left H-mod, then

$M^H = \{m\in M \mid h\cdot m = \epsilon(h).m,\ \forall h\in H\}$ $\overset{\text{basis}}{\Downarrow}$

$= \{m\in M \mid g\cdot m = \epsilon(g).m = m,\ \forall g\in G\} = M^G$

if M is a right H-comod

then $M^{\mathrm{co}H} = \{m\in M \mid \rho(m) = m\otimes 1\} = M,$

③ let $H = U(\mathfrak{g})$. If M is a left H-mod, then

$M^H = \{m\in M \mid x\cdot m = \epsilon(x)\cdot m = 0,\ \forall x\in \mathfrak{g}\}$

the constants of the action $\mathfrak{g}$.

④. Consider H as a left H-mod via left multiplication

Then $H^H = \{t\in H \mid h\cdot t = \epsilon(h)\cdot t,\ \forall h\in H\}$

Example: $H = \Bbbk G$, $\# G > 1$. $H^H = \{x\in H \mid g\cdot x = x,\ \forall g\in G\} = 0$.

The question will be considered in Cpt2. as to when $H^H \neq 0$.

Cpt1.8. tensor products of mod, comod.

recall: alg., coalg. structure of $A\otimes B$, $C\otimes D$ $\xrightarrow[m,\eta\text{ coalg map}]{\Delta,\epsilon\text{ alg map}}$ bialg.

(γ coalg map?)

now: mod, comod structure + ρ alg map $\Rightarrow$ hopf module.

1. H-mod $V\otimes W$

① let V, W be left H-mod. then $V\otimes W$ is also a left

H-mod via $\gamma: H\otimes(V\otimes W) \longrightarrow V\otimes W$

$h \otimes (v \otimes w) \mapsto (h_1 \cdot v) \otimes (h_2 \cdot w)$

i.e.

$$\begin{tikzcd} H \otimes V \otimes W \arrow[r, "\gamma"] \arrow[d, "\Delta \otimes I_{V \otimes W}"'] & V \otimes W \\ H \otimes H \otimes V \otimes W \arrow[r, "I_H \otimes \tau \otimes I_W"] & (H \otimes V) \otimes (H \otimes W) \arrow[u, "\gamma_V \otimes \gamma_W"'] \end{tikzcd}$$

② If H is cocommutative, then $V \otimes W \cong W \otimes V$ as H-mod

However, it is false in general.

Ex: let G be a finite non-abelian grp. $H = (\mathbb{K}G)$

⊗. Structure of H^*

G is a basis of $\mathbb{K}G$, let $G^* = \{g^* \mid g \in G\}$ be its dual basis

$\Delta_{H^*} = m^*$, $m_{H^*} = \Delta^*$, u_{H^*}, ϵ_{H^*} are determined by G^*

(1) $\forall\, g^*, h^* \in G^*$, $x \in G$,

Since $\langle \Delta^*(g^* \otimes h^*), x \rangle = \langle g^* \otimes h^*, \Delta(x) \rangle$

$= \langle g^* \otimes h^*, x \otimes x \rangle$

$= \delta_{xg} \cdot \delta_{hx}$

we have $\Delta^*(g^* \otimes h^*) = \delta_{gh} \cdot g^*$

(2) $\langle u_{H^*}(1), x \rangle = \langle \epsilon^*(1^*), x \rangle = \langle 1^*, \epsilon(x) \rangle = 1$

Hence $u_{H^*}(1) = \sum_{g \in G} g^*$

product ✓

unit ✓

(3). $\langle m^*(g^*), x \otimes y \rangle = \langle g^*, m(x \otimes y) \rangle$

$= \langle g^*, xy \rangle$

$= \delta_{xy, g}$

hence. $m^*(g^*) = \sum_{x \cdot y = g} x^* \otimes y^*$

(4) $\epsilon_{H^*}(g^*) = \langle u^*(g^*), 1 \rangle = \langle g^*, u(1) \rangle = \delta_{ge}$

coproduct ✓

counit ✓

let $g, h \in G$ s.t. $gh \neq hg$. then $V = \mathbb{K}.g^*$, $W = \mathbb{K}h^*$ are H^*-mods

$(gh)^* \cdot g^* \otimes h^* = \Delta_{H^*}(gh)^* \cdot g^* \otimes h^*$

$= \sum_{x \cdot y = gh} x^* \cdot g^* \otimes y^* \cdot h^* = g^* \otimes h^*$

$(gh)^* \cdot h^* \otimes g^* = \sum_{x \cdot y = gh} x^* \cdot h^* \otimes y^* \cdot g^* = 0$

2. H-comod $W \otimes V$

① let V, W be right H-comod, then $V \otimes W$ is also a right H-comod via $\rho: V \otimes W \to (V \otimes W) \otimes H$

$$v \otimes w \mapsto (v_0 \otimes w_0) \otimes v_1 w_1$$

i.e.

$$\begin{tikzcd} V \otimes W \arrow[r, "\rho"] \arrow[d, "\rho_V \otimes \rho_W"'] & V \otimes W \otimes H \\ (V \otimes H) \otimes (W \otimes H) \arrow[r, "I_V \otimes \tau \otimes I_H"] & (V \otimes W) \otimes (H \otimes H) \arrow[u, "I_{V\otimes W} \otimes m"'] \end{tikzcd}$$

Cpt 1.9. Hopf mods.

1. def

① M is a right H-Hopf mod, if the following condition hold.

(1) M is a right H-mod via $\gamma: M \otimes H \to M$

(2) M is a right H-comod via $\rho: M \to M \otimes H$

(3) ρ is a right H-mod map.

i.e. $\gamma_{M \otimes H} \circ \rho \otimes I_H = \rho \circ \gamma$

sweedler: $(m \cdot h)_0 \otimes (m \cdot h)_1 = m_0 \cdot h_1 \otimes m_1 \cdot h_2$

diagram:

$$\begin{tikzcd} \underline{M \otimes H} \arrow[r, "\rho \otimes I_H"] \arrow[d, "\gamma"'] & \underline{M \otimes H} \otimes H \arrow[d, "\gamma_{M \otimes H}"] \\ M \arrow[r, "\rho"] & M \otimes H \end{tikzcd}$$

② More generally, in the module part of the definition we may replace H by any sub Hopf algebra K of H then M becomes a right (H, K)-Hopf modules.

③ Cat: ${}^H M_K$, ${}^H_K M$, M^H_K, ${}_K M^H$

2. Examples

① $M=H,\ \rho=\Delta,\ \gamma=m \Rightarrow (M,\rho,\gamma)$ is a H-hopf mod.

② let (W,γ) be any right H-module.

then $M=W\otimes H$ is also a right H-mod.

via $\gamma': (W\otimes H)\otimes H \to W\otimes H$

$(w\otimes h)\otimes x \mapsto (w\cdot x_{(1)})\otimes(h\cdot x_{(2)})$

let $\rho = I_W\otimes\Delta$, then $(W\otimes H,\gamma',\rho)$ is a right H-hopf module.

③. As a special case of this example.

let W be the trivial H-mod. i.e. $W=W^H$

$\forall w\in W, h\in H,\ w\cdot h=\epsilon(h)\cdot w$

by ②. W is a H-mod $\Rightarrow M=W\otimes H$ is a H-hopf mod.

Specificly, $(w\otimes h)\cdot x = (w x_{(1)})\otimes h x_{(2)}$

$= \epsilon(x_{(1)})\cdot w\otimes h x_{(2)}$ H-mod

$= w\otimes h\cdot x$

$\rho(w\otimes h) = w\otimes h_{(1)}\otimes h_{(2)}$ H-comod

Such an M is called a trivial Hopf module.

Remark: M is "trivial" from the point of view of both module and comodule: $w\otimes h\cdot x = w\otimes \underline{h\cdot x} \to$ trivial product.

$\rho(w\otimes h) = w\otimes\rho(h) \to$ trivial coprod.

3. Fundamental theorem of hopf modules.

$M\in M_H^H \Rightarrow M\cong M^{coH}\otimes H$ as right H-Hopf modules

where M^{coH} is a trivial H module.

In particular, M is a free right H-mod of rank $=\dim M^{coH}$

pf: let $\alpha: M^{coH}\otimes H\to M$, $\beta: M\to M^{coH}\otimes H$

$m\otimes h\mapsto m\cdot h$ $\qquad m\mapsto m_0\cdot(Sm_1)\otimes m_2$

(1) we first show that $m_0 \cdot S(m_1) \in M^{coH}$

Since $\rho(m_0 \cdot S(m_1)) \overset{\text{H-mod map}}{=} \rho(m_0)\cdot S(m_1)$

$= (m_0 \otimes m_1)\cdot \Delta(S(m_2))$ S anti-coalg.

$= (m_0 \otimes m_1)\cdot S(m_3) \otimes S(m_2)$

$= m_0 \cdot S(m_3) \otimes m_1 \cdot S(m_2)$ $m_1 S(m_2) = \epsilon(m_1)\cdot 1$

$= m_0 \cdot S(m_2) \otimes \epsilon(m_1)\cdot 1$

$= m_0 \cdot S(\epsilon(m_1)\cdot m_2) \otimes 1$

$= m_0 \cdot S(m_1) \otimes 1$

(2) next show that $\alpha \circ \beta = I_M$, $\beta \circ \alpha = I_{M^{coH} \otimes H}$

$\alpha \circ \beta(m) = \alpha(m_0 \cdot S(m_1) \otimes m_2)$

$= m_0 \cdot S(m_1)\cdot m_2$

$= m_0 \cdot \epsilon(m_1)\cdot 1 = m$

$\beta \circ \alpha(m \otimes h) = \beta(m \cdot h)$

$= (m\cdot h)_0 \cdot S((m\cdot h)_1) \otimes (m\cdot h)_2$ ρ H-mod map

$\rho(m) = m \otimes 1$ $= m_0 h_1 \cdot S(m_1 h_2) \otimes m_2 h_3$

$= m \cdot h_1 \cdot S(h_2) \otimes h_3$

$= m \cdot \epsilon(h_1) \otimes h_2$

$= m \otimes h$

(3). finally, check that α is a right H-hopf mod map

4. Example

let $H = \mathbb{k}G$, and M be a H-hopf mod.

(1) M is an H-comod $\Rightarrow M = \bigoplus_{g\in G} M_g$, where $M_g = \{m \in M \mid \rho(m) = m \otimes g\}$

M is an H-mod $\Rightarrow G$ act on M

$\rho: M \to M \otimes H$ is an H-mod map $\Rightarrow \rho(m\cdot h) = \rho(m)\cdot h$

(2) hence $\rho(m_g \cdot h) = \rho(m_g)\cdot h$

$= (m_g \otimes g)\cdot h = (m_g \cdot h) \otimes gh$

therefore $m_g \cdot h \in M_{gh}$

(3) so the G-action permute $\{M_g\}$

in particular $M_1 \cdot g = M_g$

(4) This is precisely what the fundamental theorem says.

here $M^{coH} = M_1$ and so $M \cong M_1 \otimes \mathbb{K}G$ as $\mathbb{K}G$-hopf mod

implies $M_g \cong M_1 \otimes g$

Remark: Hopf mods are trivial but useful

The difficult is to prove that a given M is an H-hopf module, so that the fund. thm. can be applied.

Cpt 3: $\dim H < \infty \Rightarrow$ right (H,K)-hopf module is a free right K-module.