Cpt 3. Freeness

Cpt 3.3. normal basis

1. lemmas

① (1) let K be a subalgebra of A, and $I \lhd A$, then $K \cap I \lhd K$.

(2) let D be a subcoalgebra of C, and $I \lhd C$, then $I \cap D \lhd D$

pf: (2) let $\pi: D \to C/I$ s.t. $\pi(d) = d + I$, then $\ker\pi = I \cap D \lhd D$ #

② let K be a subbialgebra of B, and I a biideal of B, then

(1) $K \cap I$ is a biideal of K.

(2) $(K \cap I) B$ is a coideal of B

pf: (2) $\Delta(K\cap I \cdot B) = \Delta(K \cap I) \cdot \Delta(B) \subseteq (K \cap I \otimes K + K \otimes K \cap I) \cdot (B \otimes B)$

$\subseteq (K \cap I) B \otimes B + B \otimes (K \cap I) B$.

Note: 1. $(K \cap I) B \subseteq KB \cap IB = B \cap I = I$.

2. $(A_1 + A_2) \cdot A_3 \subseteq A_1 \cdot A_3 + A_2 \cdot A_3$

(3) $(K \cap I) B$ is a right B-module and a K-bimodule.

③ let K be a subbialgebra of B, $I = \ker \epsilon$ be the maximal biideal of B, and $I' = (K \cap I) \cdot B$. Then

(1) $B/I \cong \mathbb{K}$ as bialgebras,

(2) B/I' is a trivial left K-module.

pf: (2) We show that $k \cdot b + I' = b \cdot k + I' = \epsilon(k) \cdot b + I'$ for any $k \in K$, $b \in B$.

Since $k \cdot b - \epsilon(k) \cdot b = (k - \epsilon(k) \cdot 1_K) \cdot b$, where $k - \epsilon(k) \cdot 1_K \in \ker\epsilon \cap K$

it follows that $k \cdot b - \epsilon(k) \cdot b \in I'$; hence we complete the proof.

2. let L be a subgroup of G, then $\mathbb{K}L$ is a sub-Hopf-algebra of $\mathbb{K}G$. Let $\{x_1, \cdots, x_n\}$ be a left coset representatives of L in G,

then (1) $\mathbb{K}G$ is a free right $\mathbb{K}L$ module with $\{x_1, \cdots, x_n\}$ as its basis.

pf: $\Bbbk G = \bigoplus_{i=1}^{n} x_i \cdot \Bbbk L$, where $x_i \cdot \Bbbk L$ are cyclic right $\Bbbk L$ module. Each $x \cdot \Bbbk L$ is free since $\operatorname{ann} x_i = 0$

Note: let (C, Δ, ϵ) be a coalgebra, denote $C^+ = \operatorname{Ker} \epsilon$ (Radford).

(2) By 1. Lemma ③, $I' = (\Bbbk L)^+ G$ is a coideal of $\Bbbk G$, and $\Bbbk G / I'$ is a trivial left $\Bbbk L$-module. Thus,

$$\Bbbk L \otimes (\Bbbk G / I') \cong \Bbbk \cdot L^{(\dim_{\Bbbk} \Bbbk G / I')} \cong \Bbbk G \quad \text{as left } \Bbbk L \text{ modules.}$$

pf: It suffices to show that $\dim_{\Bbbk} \Bbbk G / I' = \frac{|G|}{|L|} = n$. By direct computation, we get $(\Bbbk L)^+ = \bigoplus_{e \neq x \in L} \Bbbk \cdot (x - e)$ and $(\Bbbk L^+) G = \sum_{g \in G} \bigoplus_{e \neq x \in L} \Bbbk \cdot (xg - g) = \bigoplus_{i=1}^{n} \bigoplus_{e \neq x \in L} \Bbbk \cdot (x \cdot x_i - x_i)$

Hence $\dim_{\Bbbk} \Bbbk G / I' = |G| - n \cdot (|L| - 1) = n$ #

Note: 1. $g - h \in (\Bbbk L)^+ G \Leftrightarrow gh^{-1} \in L \Leftrightarrow gL = hL$

2. $\{x_1, \cdots, x_n\}$ is a normal basis of $\Bbbk G$ as free $\Bbbk L$-module.

3. Theorem (Schneider)

Let H be a f.d. Hopf algebra and K be a sub-Hopf-algebra of H, then

(1). $H \cong K \otimes (H / K^+ H)$ as left K-modules and right $H / K^+ H$-comodules.

(2) $H \cong (H / HK^+) \otimes K$ as right K-modules and left H / HK^+-comodules.

Note: 1. To show that $H \cong K \otimes (H / K^+ H)$ as left K-modules, one only need to prove that $\dim K / \dim H = \dim H / K^+ H = \dim H - \dim K^+ H$.

2. Let K be a subalgebra of A, then A-modules are naturally K-modules.

Let D be a quotient coalgebra of C, then C-comodules are naturally D-comodules

pf: $\gamma : M \otimes A \to M \Rightarrow \gamma' : M \otimes K \hookrightarrow M \otimes A \to M$.

$\rho : M \to M \otimes C \Rightarrow \rho' : M \to M \otimes C \twoheadrightarrow M \otimes D$

3. The comodule part of the theorem can be proved dually. (待证).

Remark: 1. The theorem tells that H has a right (left) normal basis over K

Cpt 8 will consider more detail about normal basis.

2. Schneider's results is a corollary to a more general result

about Galois extension and crossed products. (Cpt 8)

3. Notice that K is a subbialgebra while B/K^+B is just a coalgebra and a right ideal of B. Masuoka weakens the condition to "K is a left coideal subalgebra". This time, the condition above is more dual

i.e. K: left coideal + subalgebra of B $K \hookrightarrow B \Rightarrow B$ mods are K-mods

K^+B: right ideal + coideal of B. $B \twoheadrightarrow B/K^+B \Rightarrow B$ comods are B/K^+B comods.

Cpt 3.4. adjoint action, normal structure.

Let H be a Hopf-algebra (H might not be f.d.)

1. definitions

① The left adjoint action of H on itself is given by

$(ad_l\, h)(k) = h_1 k (S(h_2))$, for all $h, k \in H$

② The right adjoint action of H on itself is given by

$(ad_r\, h)(k) = S(h_1)\cdot k \cdot h_2$, for all $h, k \in H$

③ A sub-Hopf-algebra K of H is called normal if both

$(ad_l H)(K) \subseteq K$ and $(ad_r H)(K) \subseteq K$.

Remark: 1. In the case of $H = \mathbb{k}G$ and $g \in G$, then $(ad_l\, g)(k) = gkg^{-1}$, all $k \in \mathbb{k}G$, and if $H = U(\mathfrak{g})$ and $x \in \mathfrak{g}$, then $(ad_l\, x)(k) = xk - kx$, all $k \in U(\mathfrak{g})$. Thus in these cases we get the usual classical adjoint actions.

2. $N \triangleleft G \Leftrightarrow$ every left coset of N is a right coset.

$\Leftrightarrow N = \ker f$, for some group morphism $f: G \to G'$.

Note: 1. $ad: G \to \mathrm{Inn}\, G \hookrightarrow \mathrm{Aut}\, G$ is a homomorphism of groups.

2. $ad: L \to \mathrm{Der}\, L \hookrightarrow \mathrm{End}\, L$ is a homomorphism of Lie algebras.

$\mathrm{Der}\, L = \{ f \in \mathrm{End}\, L \mid f([x,y]) = [f(x), y] + [x, f(y)] \}$

3. $ad: H \to\ ?\ \hookrightarrow\ ?$ is a homomorphism of ?

2. lemmas

① 1. If $f: A \to B$ is a algebraic map, then every B-module M has a natural A-module structure via f, i.e. $m \cdot a = m f(a)$, $\forall m \in M, a \in A$.

2. If $g: C \to D$ is a coalgebraic map, then every C-comodule M has a natural D-comodule structure via g, i.e. $\rho(m) = m_0 \otimes g(m_1)$, $\forall m \in M$.

② let M be an H-Hopf-module, then $M^{coH} = \{ m_0 S(m_1) \mid \forall m \in M \}$

③. let K be a sub-Hopf-algebra of H, and I a hopf ideal of H.

(1) If K is normal, then $HK^+ = K^+H$ is a Hopf ideal of H, and $\pi: H \to H/HK^+$ is a morphism of Hopf algebras.

(2). Let $\pi: H \to H/I = \bar{H}$ be a morphism of Hopf algebras, and consider H as an $\bar{H}$-bicomodule. Then $H^{co\bar{H}}$ is ad_l-stable and ${}^{co\bar{H}}H$ is ad_r-stable.

Note: $N \lhd G \Rightarrow a \cdot n = a n a^{-1} \cdot a \in Na$, for all $n \in N$, $a \in G$

Pf: (1). Consider the identity

$$h \cdot a = h_1 \cdot a \cdot \epsilon(h_2) = h_1 \cdot a \cdot S(h_2) h_3 = ad_l(h_1)(a) \cdot h_2 \quad \text{for all } h \in H,\ a \in K$$

Moreover, if $\epsilon(a) = 0$, then $\epsilon(ad_l(h_1)(a)) = \epsilon(h_1 \cdot a \cdot S(h_2)) = 0$, and thus $HK^+ \subseteq K^+H$. The other containment follows analogously. It follows that HK^+ is an ideal (it is always a coideal) and $S(HK^+) = S(K^+) S(H) \subseteq K^+H = I$.

Thus HK^+ is a Hopf ideal and π is a Hopf morphism.

(2). $\forall a \in H^{co\bar{H}}$, $h \in H$, $\rho(ad_l h(a)) = \rho(h_1 a S(h_2))$

$$= (h_1 a S(h_2))_0 \otimes \overline{(h_1 a S(h_2))_1}$$

$$= h_1 a_0 S(h_4) \otimes \overline{h_2 a_1 S(h_3)} \quad a_0 \otimes \bar{a}_1 = a \otimes \bar{1},\ \pi \text{ is algebraic}$$

$$= h_1 a S(h_3) \otimes \epsilon(h_2) \cdot \bar{1}$$

$$= h a \otimes \bar{1}$$

Thus $ad_l(h)(a) \in H^{co\bar{H}}$. The arguement is similar on the left.

Note: 1. let $L \lhd G$, then $\mathbb{K}L^+ \cdot G = G \cdot \mathbb{K}L^+$ and $\mathbb{K}G/\mathbb{K}L^+ \cong \mathbb{K} \cdot G/L$

let $\bar{H} = \mathbb{K}\cdot G/L$, then $\sum_i x_i g_i \in H^{\omega\bar{H}} \Leftrightarrow \bar{g_i} = \bar{1}, \forall x_i \neq 0$, hence $H^{\omega\bar{H}} = \mathbb{K}L$

2. K is a sub-Hopf-algebra $\Rightarrow K^+H$ is a wideal and right ideal of H

I is a hopf ideal $\Rightarrow H^{\omega H/I}$ is a sub-Hopf-algebra and ~~right wideal~~ of H ?

Pf: $g, h \in H^{\omega H/I} \Rightarrow \rho(g\cdot h) = \rho(g)\cdot\rho(h) = g\otimes\bar{1}\cdot h\otimes\bar{1} = gh\otimes\bar{1}$.

$\forall h \in H^{\omega H/I} \Rightarrow \rho(h_0^i)\otimes\bar{h_1^i} = h\otimes\bar{1}\otimes\bar{1} \not\Rightarrow \rho(h_0^i) = h_0^i\otimes\bar{1}$?

Remark: The converse of (1) is open in general, but is true for "nice" extensions.

3. faithfully flat

① A ring extension $A \subseteq B$ is left faithfully flat if for any right A-module map $f: M \to N$, f is injective $\Leftrightarrow f\otimes I_B : M\otimes_A B \to N\otimes_A B$ is injective.

That is, B is a flat left A-module via the extension.

② Given two maps $f, g: M \to N$, the equalizer of f and g is $\mathrm{Ker}(f,g) = \{m \in M \mid f(m) = g(m)\}$. The equalizer diagram $L \xrightarrow{h} M \overset{f}{\underset{g}{\rightrightarrows}} N$ is exact if $\mathrm{Im}\, h = \mathrm{Ker}(f,g)$ and h is injective.

Note: When $g = 0$, $L \xrightarrow{h} M \overset{f}{\underset{g}{\rightrightarrows}} N$ is exact $\Leftrightarrow 0 \to L \to M \xrightarrow{f} N \to 0$ is exact.

③. Let K be a sub-Hopf-algebra of H s.t. H is left or right faithfully flat over K, and such that $HK^+ = K^+H$. Let $\bar{H} = H/HK^+$ and consider H as an $\bar{H}$-bicomodule as before. Then

(1). $K = H^{\omega\bar{H}} = {}^{\omega\bar{H}}H$

(2) K is a normal sub-Hopf-algebra of H.

Note: 1. $HK^+ = K^+H \Rightarrow K^+H$ is a Hopf ideal of H.

2. H is left faithfully flat over K, then $H\otimes_K K, K\otimes_K H \hookrightarrow H\otimes_K H$ and

?$\Rightarrow$ $H\otimes_K K \cap K\otimes_K H = K\otimes_K K$.

Pf: (1). " let $f, g: H \to H\otimes_K H$ s.t. $f(h) = h\otimes_K 1$, $g(h) = 1\otimes_K h$, $\forall h \in H$

The diagram $K \hookrightarrow H \overset{f}{\underset{g}{\rightrightarrows}} H\otimes_K H$ is exact since $h\otimes_K 1 = 1\otimes_K h \Leftrightarrow h\otimes_K 1 \in K\otimes_K K$

2. Let $\bar{H} = H/HK^+$, then the diagram $H^{\omega\bar{H}} \hookrightarrow H \rightrightarrows H\otimes H$ is also exact, where

the two maps on the right are given by $h \mapsto h\otimes\bar{1}$, $h\mapsto h_1\otimes \bar{h}_2$.

(By definition, $h\otimes\bar{1} = h_1\otimes\bar{h}_2 \Leftrightarrow h\in H^{co\bar{H}}$)

3. Finally, we tie these two diagrams together. Define a map

$$\beta : H\otimes_K H \to H\otimes\bar{H}, \text{ via } x\otimes_K y \mapsto xy_1\otimes\bar{y}_2$$

Note: 1. this is the Galois map studied in (pt 8)

2. β is well-defined since $xy_1\otimes\bar{y}_2$ is K-bilinear for parameters x and y.

β has a well-defined inverse, namely $x\otimes\bar{y}\mapsto xSy_1\otimes_K y_2$, and thus β is bijective. It's also easy to check that $K\subseteq H^{co\bar{H}}$.

Note: 1. $\beta^{-1}\circ\beta(x\otimes_K y) = \beta^{-1}(xy_1\otimes\bar{y}_2) = xy_1Sy_2\otimes_K\bar{y}_3 = x\otimes_K y$

$\beta\circ\beta^{-1}(x\otimes\bar{y}) = \beta(xSy_1\otimes_K y_2) = xSy_1y_2\otimes\bar{y}_3 = x\otimes y$

β^{-1} is well-defined since $hk\in HK^+ \Rightarrow \beta^{-1}(x\otimes hk) = 0$?

2. $k\in K \Rightarrow \rho(k) = k_1\otimes\bar{k}_2 = k_1\otimes\overline{(k_2-\epsilon(k_2)1_K)+\epsilon(k_2)1_H} = k_1\otimes\epsilon(k_2)\cdot\bar{1} = k\otimes\bar{1}$

Thus we have a commutative diagram :

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "i"'] & H \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & H\otimes_K H \arrow[d, "\beta", "\wr"'] \\ H^{co\bar{H}} \arrow[r, hook] & H \arrow[r, shift left] \arrow[r, shift right] & H\otimes\bar{H} \end{tikzcd}$$

(Five lemma)

By exactness and the bijectivity of β, we must have $K = H^{co\bar{H}}$.

Using $\bar{H}\otimes H$ and repeating the arguement, we obtain $K = {}^{co\bar{H}}H$.

(2) It follows from 2③.

Corollary: Let H be f.d. and K a sub-Hopf-algebra, Then K is normal iff $HK^+ = K^+H$.

pf: H is free over K $\Rightarrow$ H is faithfully flat.

Note: Free modules and projective modules are flat modules.

Remark: 1. Cpt4: when H is faithfully flat over K.

2. A more difficult question: when Hopf ideals are of the form $HK^+ = K^+H$.

4. adjoint coaction

① definitions

(1) The left adjoint coaction of H on itself is given by

$$\rho_\ell : H \to H\otimes H \quad \text{via} \quad h \mapsto h_1 S h_3 \otimes h_2$$

(2) The right adjoint coaction of H on itself is given by

$$\rho_r : H \to H\otimes H \quad \text{via} \quad h \mapsto h_2 \otimes (Sh_1) h_3$$

(3) A Hopf ideal I of H is called normal if both

$$\rho_\ell(I) \subseteq H\otimes I \quad \text{and} \quad \rho_r(I) \subseteq I \otimes H$$

(that is, I is a subcomodule of H under ρ_ℓ and ρ_r)

If I is normal, $\pi : H \to H/I$ is called conormal.

② some dual proprities

(1). H is a left H-module via left adjoint action.

pf: $ad_\ell g \circ ad_\ell h (k) = ad_\ell g (h_1 k S(h_2)) = g_1 h_1 k S(h_2) S(g_2) = (gh)_1 k S(gh)_2 = ad_\ell gh (k)$

$ad_\ell 1_H (k) = 1_1 k \cdot S(1_2) = k$,

Note: In Cpt2, H is a left H-module via $h \rightharpoonup H = H\cdot S(h)$

(2) H is a right H-comodule via right adjoint coaction.

pf: $\rho_r \otimes I_H \circ \rho_r(h) = h_3 \otimes (Sh_2) h_4 \otimes (Sh_1) h_5 = I_H \otimes \Delta \circ \rho_r(h)$

$(I_H \otimes \epsilon)(h_2 \otimes (Sh_1) h_3) = h_2 \cdot \epsilon(h_1) \cdot \epsilon(h_3) = h$

(3) H is commutative $\Rightarrow$ H is a trivial left module via ad_ℓ.

H is cocommutative $\Rightarrow$ H is a trivial right comodule via ρ_r.

pf: $ad_\ell(h)(k) = h_1 k S(h_2) = k \cdot h_1 S(h_2) = \epsilon(h)\cdot k$, $\forall k \in K,\ h \in H$

$\rho_r(h) = h_2 \otimes (Sh_1) h_3 = h_1 \otimes (Sh_2) h_3 = h \otimes 1$, $\forall h \in H$.

(4). diagrams

$$\begin{tikzcd}
H\otimes H \arrow[rr] \arrow[d, "\Delta\otimes 1_H"'] & & H \\
H\otimes H\otimes H \arrow[dr, "1_H\otimes\tau"'] & & H\otimes H\otimes H \arrow[u, "m^2"'] \\
& H\otimes H\otimes H \arrow[ur, "1_H^2\otimes S"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
H\otimes H & & H \arrow[ll] \arrow[d, "\Delta^2"] \\
H\otimes H\otimes H \arrow[u, "1_H\otimes m"] & & H\otimes H\otimes H \arrow[dl, "S\otimes 1_H^2"] \\
& H\otimes H\otimes H \arrow[ul, "\tau\otimes 1_H"] &
\end{tikzcd}$$

$ad_\ell = m^2 \circ I_H^2 \otimes S \circ I_H \otimes \tau \circ \Delta \otimes I_H$ $\qquad$ $\rho_r = I_H \otimes m \circ \tau \otimes I_H \circ S \otimes I_H^2 \circ \Delta^2$

$(ad_\ell\, h)(k) = h_1 k (S(h_2))$ $\qquad$ $\rho_r(h) = h_2 \otimes (Sh_1) h_3$

(5).

recall: $\pi : H \to H/I$ is a surjective morphism of Hopf algebras.

$i : K \hookrightarrow H$ is a injective morphism of Hopf algebras.

I.

K is ad_ℓ-stable $\Leftrightarrow$ $ad_\ell(H)(K) \subseteq K$, i.e. K is a left H-submodue via ad_ℓ

$\Leftrightarrow$ $\psi = ad_\ell \circ (I_H \otimes i)$ factors through $i : K \hookrightarrow H$

that is $\exists f : H \otimes K \to K$ s.t.

$$\begin{tikzcd} H\otimes K \arrow[r, hook, "I_H\otimes i"] \arrow[dr, "f"'] & H\otimes H \arrow[r, "ad_\ell"] \arrow[d, dashed, "ad_\ell|_{H\otimes K}"] & H \\ & K \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative.

Here $f(h \otimes k) = h_1 k\, Sh_2$, $\forall\, k \in K,\ h \in H$. ($f$ is well-defined iff $ad_\ell(H\otimes K) \subseteq K$)

II. I is ρ_r-stable $\Leftrightarrow$ $\rho_r(I) \subseteq I \otimes H$, i.e. I is a right H-comodule via ρ_r.

$\Leftrightarrow$ $\psi = (\pi \otimes I_H) \circ \rho_r$ factors through $\pi : H \to H/I$

that is $\exists\, g : H/I \to H/I \otimes H$, s.t.

$$\begin{tikzcd} H \arrow[r, "\rho_r"] \arrow[dr, "\pi"'] & H\otimes H \arrow[r, "\pi\otimes I_H"] & H/I\otimes H \\ & H/I \arrow[ur, "g"'] & \end{tikzcd}$$

is commutative.

Here $g(\bar{h}) = \bar{h}_2 \otimes (Sh_1)h_3$, $\forall\, h \in H$. (g is well-defined iff $\rho_r(I) \subseteq I \otimes H$)

Corollary: If H is f.d. then I is a ρ_r-stable Hopf ideal of H

$\Leftrightarrow$

$$\begin{tikzcd} H \arrow[r, "\rho_r"] \arrow[dr, "\pi"'] & H\otimes H \arrow[r, "\pi\otimes I_H"] & H/I\otimes H \\ & H/I \arrow[ur, "g"'] & \end{tikzcd}$$

is commutative.

$\Leftrightarrow$

$$\begin{tikzcd} H^* & H^*\otimes H^* \arrow[l, "\rho_r^*"'] & \underbrace{(H/I)^*\otimes H^*}_{?} \arrow[l, "\pi^*\otimes I_{H^*}"'] \arrow[dl, "g^*"] \\ & (H/I)^* \arrow[ul, "\pi^*"] & \end{tikzcd}$$

is commutative.

$\Leftrightarrow$ $(H/I)^*$ is an ad_r-stable subHopf algebra of H^*

Note: $\rho_\ell(I) \subseteq H \otimes I \Rightarrow ad_\ell(H^*)(H/I)^* \subseteq (H/I)^*$

pf: $\forall\, h^* \in H^*$, $\bar{k}^* \in (H/I)^*$, we show that $h_1^* \bar{k}^* S^* h_2^* (I) = 0$

$\forall\, x \in I$, $\langle h_1^* \bar{k}^* S^* h_2^*, x \rangle = \langle m_{H^*}^2 (h_1^* \otimes \bar{k}^* \otimes S^* h_2^*), x \rangle$

$= \langle h_1^* \otimes \bar{k}^* \otimes S^* h_2^*, x_1 \otimes x_2 \otimes x_3 \rangle$

$= h_1^*(x_1) \cdot \bar{k}^*(x_2) \cdot h_2^*(S x_3)$

$= \langle \triangleleft_{H^*}(h^*), x_1 \otimes S x_3 \rangle \cdot \bar{k}^*(x_2)$

$= h^*(x_1 S x_3) \cdot \bar{k}^*(x_2)$

$= \langle h^* \otimes \bar{k}^*, x_1 S x_3 \otimes x_2 \rangle$

Since $\rho_l(I) \subseteq H \otimes I$, we have $\bar{k}^*(x_2) = 0$. #

Q: 这里有些奇怪 left → left ?

比如，$(V \otimes W)^*$ 对偶应该是 $W^* \otimes V^*$ 才对吗？

5. some speculation

① Normal hopf ideal also arise in the text of affine $\mathbb{k}$-groups

② let $\varphi(K) = HK^+$, $\psi(I) = {}^{coH/I}H$

then $\left\{ \begin{array}{l} K \text{ a normal} \\ \text{subHopf-algebra of } H \end{array} \right\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \left\{ \begin{array}{l} I \text{ a normal} \\ \text{Hopf ideal of } H \end{array} \right\}$

Note: It's trivial that HK^+ is a left H-module and that $H^{co\bar{H}}$ is ad_l-stable. However, it's not easy to show that HK^+ is ρ_l or ρ_r-stable, $H^{co\bar{H}}$ is a left or right H-comodule.

③ φ and ψ are inverse bijections if either H is commutative or if the coradical H_0 of H is cocommutative.

Cpt 3.5. faithful freeness.

1. lemma: let $K \subseteq E$ be a Galois field extension with Galois group G, and let H be a Hopf algebra over E. Assume that G acts on H as semilinear automorphism, Then H^G is a Hopf algebra over $\mathbb{K}$.

pf: pass.

2. let $F \subseteq E$ be a Galois field extension of degree 2, with Galois group $\{1, \sigma\}$, let σ act on $\mathbb{Z}$ by $z \mapsto -z$. Then G acts on the group algebra $E\mathbb{Z}$ by acting on both E and $\mathbb{Z}$. Let $H = (E\mathbb{Z})^G$ and

$K = (E(n\mathbb{Z}))^G \subseteq H$. If n is even, then H is not free over K.

Pf: pass.

Note: Though H is not a free K-module, it still might be faithfully flat.

3. H is free over the f.d. sub-Hopf-algebra K if

(1). K is s.s.

(2). K is normal.

4. Conjecture.

Is H always left and right faithfully flat over K?