Cpt 7. crossed product

Cpt 7.3 equivalent class

1. Inner action.

H measures A $\begin{cases} h\cdot ab = (h_1\cdot a)(h_2\cdot b) \\ h\cdot 1_A = \varepsilon(h)\cdot 1_A \end{cases}$

The measuring is inner if there exists a convolution invertible map $u \in \mathrm{Hom}(H, A)$, such that for all $h \in C$, $b \in B$

$$h\cdot b = u(h_1)\, b\, u^{-1}(h_2)$$

2. twisted product

① H measures A, $\sigma \in \mathrm{Hom}(H\otimes H, A)$ convolution invertible

(1) $\Rightarrow$ $A \#_\sigma H$ is a crossed product

$$(a\# h)(b\# k) = a(h_1\cdot b)\,\sigma(h_2, k_1)\,\# h_3 k_2$$

(2) trivial $\sigma \Rightarrow$ smash product $A\# H$

$$(a\# h)(b\# k) = a(h_1\cdot b)\,\# h_2 k$$

(3) trivial measuring $\Rightarrow$ twisted product $A_\sigma[H]$

$$(a\otimes h)(b\otimes k) = ab\,\sigma(h_1, k_1)\otimes h_2 k_2$$

② Let $A\#_\sigma H$ be a crossed product s.t. the measure of H on A is inner, via some invertible $u \in \mathrm{Hom}(H, A)$

Define $\tau : H\otimes H \to A$

$$h\otimes k \mapsto u^{-1}(k_1)\, u^{-1}(h_1)\, \sigma(h_2, k_2)\, u(h_3 k_3)$$

Then τ is a cocycle and $A\#_\sigma H \cong A_\tau[H]$, a twisted product with trivial action, via

$\phi : A\#_\sigma H \to A_\tau[H]$, $\quad \psi : A_\tau[H] \to A\#_\sigma H$

$a\# h \mapsto a\, u(h_1)\otimes h_2 \qquad a\otimes h \mapsto a\, u^{-1}(h_1)\# h_2$

i.e. (1) $\phi \circ \psi = I_{A_\tau[H]}$, (2) $\psi \circ \phi = I_{A \#_\sigma H}$

ϕ is an algebra map.

Moreover, ϕ is both a left A-mod and a right H-comod map (trivial)

pf: (1). $\psi \circ \phi (a \# h) = \psi(a u(h_1) \otimes h_2) = a u(h_1) u^{-1}(h_2) \# h_3 = a \# h$

$\phi \circ \psi (a \otimes h) = \phi(a u^{-1}(h_1) \# h_2) = a u^{-1}(h_1) u(h_2) \otimes h_3 = a \otimes h$

(2)

$\phi((a\#h)(b\#k)) = \phi(a(h_1 \cdot b)\sigma(h_2, k_1) \# h_3 k_2)$

$= a(h_1 \cdot b)\sigma(h_2,k_1) u((h_3 k_2)_1) \otimes (h_3 k_2)_2$

$= a(h_1 \cdot b)\underline{\sigma(h_2,k_1) u(h_3 k_2)} \otimes h_4 k_3$

$\downarrow \tau(h,k) = u^{-1}(k_1) u^{-1}(h_1) \sigma(h_2,k_2) u(h_3 k_3)$

$= a(h_1 \cdot b) u(h_2) u(k_1) \tau(h_3, k_2) \otimes h_4 k_3$

$\downarrow h \cdot b = u(h_1) b u^{-1}(h_2)$

$= \underline{a u(h_1)}\, \underline{b u(k_1)} \tau(h_2, k_2) \otimes h_3 k_3$

$= (a u(h_1) \otimes h_2) \cdot (b u(k_1) \otimes k_2)$

$= \phi(a\#h)\phi(b\#k)$

③. example.

1. Let $A\#H$ be a smash product s.t. the H-action is inner via $u \in Alg(H, A)$; in this case, we say that the action is strongly inner.

Then $A\#H \cong A \otimes H$, for the cocycle τ becomes trivial.

Note: smash product $\Rightarrow$ σ trivial

$\Rightarrow \tau(h,k) = u^{-1}(k_1) u^{-1}(h_1) u(h_2 k_2)$

2. In particular, let H act on itself via the left adjoint action. Then $H\#H \cong H \otimes H$. (Cpt 4)

Note: $h \cdot k = ad_l\, h\, k = h_1 k S h_2 \Rightarrow u : H \to H$ is an algebra map

$h \mapsto h$

3. Main theorem

(1) Let A be an algebra and H be a Hopf algebra, with two crossed product actions $h\otimes a \mapsto h\cdot a$, $h\otimes a \mapsto h\cdot' a$ with respect to two cocycles $\sigma, \sigma' : H\otimes H \to A$, respectively.

(2) Assume that $\phi : A\#_\sigma H \to A\#_{\sigma'} H$ is an algebra isomorphism, which is also a left A-module, right H-comodule map.

Then there exists a convolution invertible map $u \in Hom(H, A)$ s.t. for all $a\in A$, $h, k \in H$,

1) $\phi(a\#h) = a u(h_1) \#' h_2$

2) $h\cdot' a = u^{-1}(h_1)(h_2\cdot a)u(h_3)$

3) $\sigma'(h,k) = u^{-1}(h_1)(h_2\cdot u^{-1}(k_1))\sigma(h_3, k_2)u(h_4, k_3)$

(3) Conversely, given a map $u\in Hom(H,A)$ such that 2) and 3) hold, then the map ϕ in 1) is an isomorphism.

pf: $\Rightarrow$

(1).

Define $u :$
$$\begin{tikzcd} H \arrow[r] \arrow[d] & A \\ 1\#_\sigma H \arrow[r, "\phi"] & A\#'_{\sigma'} H \arrow[u, "I_A\otimes\varepsilon"'] \end{tikzcd}$$
, i.e. $u(h) = (I_A\otimes\varepsilon)\circ\phi(1\#h)$

$\phi(a\#h) = a u(h_1)\#' h_2 \Leftrightarrow \phi(1\#h) = u(h_1)\#' h_2$, since ϕ is an A-module map. It's enought to show that the right part holds.

Since ϕ is a right H-comodule map.

$$(I_A\otimes\Delta)\circ\phi(a\#h) = (\phi\otimes 1_H)\circ(I_A\otimes\Delta)(a\#h), \quad \forall a\in A,\ h\in H.$$

$\Downarrow$ applying $I_A\otimes\varepsilon\otimes I_H$ to both sides

$$\phi(a\#h) = (I_A\otimes\varepsilon\otimes I_H)\circ(\phi\otimes 1_H)\circ(I_A\otimes\Delta)(a\#h)$$
$$= (I_A\otimes\varepsilon\otimes I_H)\circ(\phi\otimes 1_H)(a\#h_1\otimes h_2)$$
$$= (I_A\otimes\varepsilon)\circ\phi(a\#h_1)\otimes h_2$$

hence $\phi(1\#h) = (I_A\otimes\varepsilon)\circ\phi(1\#h_1)\otimes h_2 \overset{A\#H \cong A\#(k\otimes H)}{=} u(h_1)\#' h_2$

(2) Similarly, as $\phi^{-1}: A\#'_{\sigma'} H \to A\#_\sigma H$ is an homomorphism satisfying the same hypotheses as ϕ, we may set $v(h) = (I_A \otimes \epsilon)\circ\phi^{-1}(1\# h)$ and conclude as above that $\phi^{-1}(a\#' h) = a\, v(h_1) \# h_2$. We claim that $v = u^{-1}$. For,

$$1\# h = \phi^{-1}\phi(1\# h) = \phi^{-1}(u(h_1)\#' h_2) = u(h_1)\, v(h_2) \# h_3$$

Applying $I_A\otimes\epsilon$ on both sides, we see that $u(h_1)v(h_2) = \epsilon(h) 1_A$

Similarly, we see $v(h_1)u(h_2) = \epsilon(h) 1_A$, and thus $u = v^{-1}$.

(3) Since ϕ^{-1} is an algebra map, we have

$$\phi^{-1}((a\#' h)(b\#' k)) = a(h_1 \cdot' b)\sigma'(h_2, k_1)\, v(h_3 k_2) \# h_4 k_3$$

$$\shortparallel$$

$$\phi^{-1}(a\#' h)\,\phi^{-1}(b\# k) = (a\, v(h_1) \# h_2)(b\, v(k_1) \# k_2)$$

$$= a\, v(h)(h_2 \cdot b\, v(k_1))\,\sigma(h_3, k_2) \# h_4 k_3$$

☆

Setting $a=b=1$, we see that

$$\sigma'(h_1, k_1)\, v(h_2 k_2) \# h_3 k_3 = v(h)(h_2\cdot v(k_1))\,\sigma(h_3, k_2) \# h_4 k_3$$

$\Downarrow$ Applying $I_A\otimes\epsilon$ on both sides

$$\sigma'(h_1, k_1)\, v(h_2 k_2) = v(h)(h_2\cdot v(k_1))\,\sigma(h_3, k_2)$$

$\Downarrow$ multiplying $*u$ on both side

$$\sigma'(h, k) = u^{-1}(h_1)(h_2\cdot u^{-1}(k_1))\,\sigma(h_3, k_2)\, u(h_4 k_3)$$

Setting $a=1$ and $k=1$, we get $\quad u(1) = (I_A\otimes\epsilon)\phi(1\# 1) = 1_A$

$$(h_1\cdot' b)\, v(h_2) \# h_3 = v(h)(h_2\cdot b) \# h_3$$

$\Downarrow$ Applying $I_A\otimes\epsilon$ on both sides

$$(h_1\cdot' b)\, v(h_2) = v(h_1)(h_2\cdot b)$$

$$\Downarrow$$

$$(h\cdot' b) = u^{-1}(h_1)(h_2\cdot b)\, u(h_3)$$

$\Leftarrow$: Let $\phi: A\#_\sigma H \to A\#'_{\sigma'} H$, $\quad \psi: A\#'_{\sigma'} H \to A\#_\sigma H$

$$a\# h \mapsto a\, u(h_1)\# h_2 \qquad\qquad a\#' h \mapsto a\, u^{-1}(h_1)\# h_2$$

The proof is similar as 7.3.1.

Note: ① $\phi: A\#_\sigma H \to A\#'_{\sigma'} H$: algebra isomorphism, left A-mod, right H-comod.

② (1) $u = (I_A \otimes \epsilon)\circ \phi$, $\phi(a\#h) = a\, u(h_1)\#' h_2$

(2) $h\cdot' a = u^{-1}(h_1)(h_2\cdot a)\, u(h_3)$

(3) $\sigma'(h,k) = u^{-1}(h_1)(h_2\cdot u^{-1}(k_1))\,\sigma(h_3,k_2)\,u(h_4,k_3)$

Note 2: $A\#_\sigma H$ may not be associative with unit.

4. Corollary (H-cleft)

Let $A \subseteq B$ be right H-cleft via γ, γ' with $\gamma(1)=\gamma'(1)=1$

Let $A\#_\sigma H$ and $A\#'_{\sigma'} H$ be the two representations of B as crossed product over A with H, with two actions and cocycles σ, σ' as described in 7.2.3 and define $u = \gamma * (\gamma')^{-1} \in \mathrm{Hom}(H, B)$

Then the actions and cocycles are related as in 7.3.4 2) and 3).

Note: 7.2.3: $h\cdot a = \gamma(h_1)\, a\, \gamma^{-1}(h_2)$, $\forall a\in A,\ h\in H$

$$\sigma(h,k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2), \quad \forall h,k\in H$$

pf: Let $\Phi: A\#_\sigma H \to B$, $\Phi': A\#'_{\sigma'} H \to B$

$a\#h \mapsto a\gamma(h)$ $\qquad$ $a\#'h \mapsto a\gamma'(h)$

Let $\theta = (\Phi')^{-1}\Phi : A\#_\sigma H \to A\#'_{\sigma'} H$,

then θ is a left A-module, right H-comodule and algebraic map. Thus, $\exists\, u \in \mathrm{Hom}(H,A)$ s.t. $\theta(a\#h) = a\,u(h_1)\#' h_2$

Applying Φ^{-1} on both sides, we see that

$\Phi(a\#h) = \Phi'(a\,u(h_1)\#' h_2)$ i.e. $a\gamma(h) = a\,u(h_1)\gamma'(h_2)$

Setting $a=1$ gives $\gamma = u*\gamma'$.

Corollary: If $A\subseteq B$ is H-cleft, then the crossed product of A by H induced by B is unique.

5. equivalent class

Let A be an algebra and H be a Hopf algebra. Two crossed product $A\#_\sigma H$ and $A\#_{\sigma'} H$ are equivalent if there exists an algebra isomorphism $\phi: A\#_\sigma H \to A\#_{\sigma'} H$ which is a left A-module and right H-comodule morphism.

Remark:

1. H cocomm. A comm: (hence A is an H-module algebra)

$\left\{ \begin{array}{l} \text{equivalent classes of } H\text{-cleft} \\ \text{extensions of } A \text{ by } H \end{array} \right\} \xrightarrow{\sim}$ the second cohomology group $\mathcal{H}^2(H, A)$

2. H cocomm, $Z(A) \hookrightarrow A$ is an H-module algebra

(略) $\leftarrow$ $\left\{ \begin{array}{l} \text{equivalent classes of } H\text{-cleft} \\ \text{extensions of } A \text{ by } H \end{array} \right\} \xrightarrow{\sim}$ the second cohomology group $\mathcal{H}^2(H, Z(A))$

Cpt 7.4. Generalized Maschke theorem and semiprime crossed product

1. recall: $\gamma: H \to 1\#H$, $h \mapsto 1\#h$, $\quad h\cdot a = \gamma(h_1)\, a\, \gamma^{-1}(h_2)$

$\sigma(h,k) = \gamma(h_1)\gamma(k_1)\gamma^{-1}(h_2k_2)$

2. Let $A\#_\sigma H$ be a crossed product where H is f.d. semisimple.

(1) Let $V \in {}_{A\#_\sigma H}\mathcal{M}$. If $W \subseteq V$ is a submodule which has a complement in ${}_A\mathcal{M}$, then W has a complement in ${}_{A\#_\sigma H}\mathcal{M}$.

(2) If A is semisimple, then so is $A\#_\sigma H$.

Note: 1. 2.2.1: $W \subseteq V$ submodule, $\pi: V \to W$ $\Bbbk$-projection

define $\tilde{\pi}: V \to W$ averaging function

$m \mapsto t_1 \cdot \pi(St_2 \cdot m)$

2. $\int_H^r = \{ t \in H \mid t\cdot h = \varepsilon(h)t,\ \forall h \in H \}$

Pf: Clearly (2) follows from (1).

Let $\pi: V \to W$ be an A-projection and choose $t \in \int_H^r$ with $\varepsilon(t) = 1$. Define $\tilde{\pi}: V \to W$

$$v \mapsto \gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)\cdot v)$$

$\forall\ w \in W$, $\tilde{\pi}(w) = \gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)\cdot w)$

$$= \gamma^{-1}(t_1)\gamma(t_2)\cdot w = w$$

hence $\tilde{\pi}^2 = \tilde{\pi}$, $V = \operatorname{Ker}\tilde{\pi} \oplus \operatorname{Im}\tilde{\pi} = \operatorname{Ker}\tilde{\pi} \oplus W$ (Fitting)

It's enough to show that $\tilde{\pi}$ is an $A\#_\sigma H$ module.

(1). $\tilde{\pi}(a\cdot v) = \gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)a\cdot v)$

$$= \gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)a\gamma^{-1}(t_3)\gamma(t_4)\cdot v) \quad \text{mul} \to \text{measuring}$$

$$= \gamma^{-1}(t_1)\cdot \pi((t_2\cdot a)\gamma(t_3)\cdot v)$$

$$= \gamma^{-1}(t_1)(t_2\cdot a)\cdot \pi(\gamma(t_3)\cdot v) \quad \text{measuring} \to \text{mul}$$

$$= \gamma^{-1}(t_1)\gamma(t_2)a\gamma^{-1}(t_3)\cdot \pi(\gamma(t_4)\cdot v)$$

$$= a\gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)\cdot v)$$

$$= a\,\tilde{\pi}(v)$$

(2). $\tilde{\pi}(\gamma(h)\cdot v) = \gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)\gamma(h)\cdot v)$ mul $\to$ 2-cocycle.

$$= \gamma^{-1}(t_1)\cdot \pi(\sigma(t_2,h_1)\gamma(t_3h_2)\cdot v)$$

$$= \gamma^{-1}(t_1)\sigma(t_2,h_1)\cdot \pi(\gamma(t_3h_2)\cdot v) \quad \text{2-cocycle} \to \text{mul}$$

$$= \gamma(h_1)\gamma^{-1}(t_1h_2)\cdot \pi(\gamma(t_2h_3)\cdot v) \quad t\cdot h = \varepsilon(h)t$$

$$= \gamma(h_1)\gamma^{-1}(t_1)\cdot \pi(\gamma(t_2)\cdot v)$$

$$= \gamma(h)\,\tilde{\pi}(v)$$

Corollary: H f.d. s.s., $V \in {}_{A\#_\sigma H}\mathcal{M}$. If V is a completely reducible A-mod, it's also completely reducible as $A\#_\sigma H$-mod. The converse is not true

3. semiprimitive

① A is semiprimitive if $\operatorname{Jac}(A) = 0$; equivalently, the intersection of primitive ideals is 0.

Note: 1. semiprimitive + Artinian $\Rightarrow$ semisimple.

2. An algebra is semiprimitive $\Leftrightarrow \exists$ a faithfully completely reducible A-mod.

②. Let H be f.d. and semisimple, and $A\#_\sigma H$ a crossed product with A semiprime. Then $A\#_\sigma H$ is semiprimitive if the H-action on A is inner.

Pf: By 7.3.1, $A\#_\sigma H \cong A_\tau[H]$, since the action is inner.

A is semiprimitive $\Rightarrow \exists\, V$ a faithfully completely reducible A-mod.

Let $\bar{V} = A_\tau[H]\otimes_A V$. Since $A_\tau[H]$ is a free left and right A-module and is an A-bimodule, we have $\bar{V} \cong V^{(n)}$ as left A modules. Since V is a completely reducible A-mod, so is $\bar{V}$. By 2. corollary, $\bar{V}$ is a completely reducible $A\#_\sigma H$-module.

It's trivial that $A_\tau[H]\otimes_A V$ is faithful.

Thus, $A_\tau[H]$ is semiprimitive

Remark: The argument does not extend to $A\#_\sigma H$ when the action of H on A is not inner.

e.g. V is completely reducible $\not\Rightarrow$ $\bar{V}$ is completely reducible

$|G|<\infty$, $A=\mathbb{k}G$, $H=(\mathbb{k}G)^*$, $V\cong \mathbb{k}$ as trivial G-module.

$\bar{V} = (A\# H)\otimes_A V$, where H act on A via $\rightharpoonup$

Note: 1. structure of $(\mathbb{k}G)^* = H$

$$\Delta(P_x) = \sum_{u\cdot v = x} P_u\otimes P_v, \quad \varepsilon(P_x) = \delta_{xe}$$

$$m(P_x P_y) = \delta_{xy} P_x, \quad u(1) = \sum_{x\in G} P_x$$

2. $h^* \rightharpoonup x = h^*(x_2)\, x_1$

Let $\{P_x \mid x\in G\}$ be a basis of H, then we have

$$\begin{aligned}(1\# P_y)(x\#1) &= ((P_y)_1 \rightharpoonup x)\#(P_y)_2\\ &= \sum_{uv=y}(P_u\rightharpoonup x)\# P_v\\ &= x\# P_{x^{-1}y}\end{aligned}$$

which shows that $A\# H$ is a free right A-module with basis

$\{1 \# P_x \mid x \in G\}$. We may identify $\overline{V} = (A \# H) \otimes_A V$ with H via $P_x \leftrightarrow (1 \# P_x) \otimes_A 1$

Now $(x \# 1)(1 \# P_y) \otimes_A 1 = (x \# P_y) \otimes_A 1$

$$= x \# P_{x^{-1} \cdot xy} \otimes_A 1$$

$$= (1 \# P_{xy})(x \# 1) \otimes_A 1$$

$$= 1 \# P_{xy} \otimes_A 1$$

which means $x \cdot P_y = P_{xy}$.

This shows that as an A-module, $\overline{V}$ is isomorphic to A under the left regular representation of A on itself.

Now if $\mathrm{char}\, \Bbbk \nmid |G|$, $\Bbbk G$ is not completely reducible Σ.

Cpt 7.5. twisted H-comod alg.

1. Let A be a left H-comodule alg and let $\sigma : H \otimes H \to \Bbbk$ s.t. σ is convolution invertible and

$$\underset{(h\cdot)}{\sigma}(k_1, m_1)\sigma(h_2, k_2 m_2) = \sigma(h_1, k_1)\sigma(h_2 k_2, m)$$

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h),$$

$\forall\ h, k, m \in H$

The σ-twisted comod alg ${}_\sigma A$ is the set A as vector space, with elements written as $\bar{a}$, for each $a \in A$, and with multiplication

$$\bar{a} \cdot \bar{b} = \underbrace{\sigma(a_{-1}, b_{-1})}_{\text{扭部分}}\, \overline{a_0 b_0}$$

Similarly, we can define right twisted comod alg, where σ is a right cocycle:

$$\sigma(k_2, m_2)\sigma(h, k_1 m_1) = \sigma(h_2, k_2)\sigma(h_1 k_1, m) \quad (\text{cop 打开})$$

2. examples.

7.5.2 EXAMPLE. Set $A = H$ and use $\Delta : H \to H \otimes H$; then we may construct ${}_\sigma H$ and H_σ. This is done in [Lu], and applied to the dual of the Drinfeld double. See §10.3. These twisted Hopf algebras generalize the notion of twisted group algebras.

Note: $\epsilon = \sigma \Rightarrow$ trivial twist

7.5.3 EXAMPLE. Let $H = \mathbf{k}G$, so that A is a G-graded algebra. For homogeneous elements $a \in A_x$ and $b \in A_y$, where $x, y \in G$, the multiplication in ${}_\sigma A$ is given by

$$\bar{a} \cdot \bar{b} = \sigma(x, y)\overline{ab}.$$

This is an example of the "cocycle twist" used in [AST]. In fact they use a bigraded algebra A and form the double twist ${}_\sigma A_{\sigma^{-1}}$. That is, A is graded by G on the left and is also graded by G on the right, so that both of these twists make sense separately.